

NAKAYAMA AUTOMORPHISM AND APPLICATIONS

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ABSTRACT. Nakayama automorphism is used to study group actions and Hopf algebra actions on Artin-Schelter regular algebras of global dimension three.

0. INTRODUCTION

Throughout let k be an algebraically closed field of characteristic zero. All algebras and Hopf algebras are over k .

The main motivation comes from noncommutative invariant theory. In a recent paper [CKWZ], Chan-Kirkman-Walton-Zhang classified and studied finite dimensional Hopf algebra actions on Artin-Schelter regular algebras of global dimension two with trivial homological determinant. One ultimate goal is to carry out the same project for Artin-Schelter regular algebras of global dimension three:

Question 0.1. [CKWZ, Question 8.1]. Let A be an Artin-Schelter regular algebra of global dimension 3. What are the finite dimensional Hopf algebras that act inner faithfully on and preserve the grading of A with trivial homological determinant?

Keeping in mind this ultimate goal, let us ask some interesting and closely related questions that make sense even in higher dimensional cases.

Question 0.2. Let A be an Artin-Schelter regular algebra and let H be a finite dimensional Hopf algebra acting on A inner faithfully.

- (1) Under what hypotheses on A , must H be semisimple?
- (2) Under what hypotheses on A , must H be a group algebra?
- (3) Under what hypotheses on A , must H be the dual of a group algebra (namely, its dual Hopf algebra H° is a group algebra)?
- (4) Assuming H is semisimple, under what hypotheses on A , must H be a group algebra (respectively, the dual of a group algebra)?
- (5) Under what hypotheses on A , does every H -action preserve the grading of A ? Assume, further, that H is a group algebra (respectively, the dual of a group algebra). Under what hypotheses on A , does the H -action preserve the grading of A ?

Question 0.2(2,4) was partially answered when A is a commutative domain by Etingof-Walton in [EW, Theorem 1.3] and when A is a skew polynomial ring with generic parameters in [CWZ, Theorems 0.4 and 4.3]. These questions arise in other subjects such as theory of infinite dimensional Hopf algebras, study of braided Hopf algebras and Nichols algebras, study of skew (or twisted) Calabi-Yau algebras, noncommutative algebraic geometry and representation theory of quivers. Another

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interesting question is [EW, Question 5.9]: If H is semisimple and A is PI (namely, A satisfies a polynomial identity), must then $\text{PIdeg}(H^\circ) \leq \text{PIdeg}(A)^2$? This paper answers some of these questions in special cases.

One basic idea (or method) in this paper is the homological identity given in [CWZ, Theorem 0.1] which roughly says that the Nakayama automorphism controls, in some aspects, the class of Hopf algebras that act on a given Artin-Schelter (or AS, for short) regular algebra A . If we know the Nakayama automorphism of A , then we can grab information about this class of Hopf algebras. Examples of such results are given in [CWZ, Theorems 0.4 and 0.6]. One immediate consequence of [CWZ, Theorem 0.6] is the following.

Corollary 0.3. [CWZ, Comment after Theorem 0.6] *Let A be the 3-dimensional Sklyanin algebra and let H be a finite dimensional Hopf algebra acting on A inner faithfully with trivial homological determinant. Then H is semisimple.*

One interesting question is to classify all possible Hopf algebras that act inner faithfully on a 3-dimensional (PI) Sklyanin algebra. Corollary 0.3 holds because the Nakayama automorphism of the 3-dimensional Sklyanin algebra is the identity map, which satisfies the hypotheses of [CWZ, Theorem 0.6]. For this reason, it is important to understand and describe explicitly the Nakayama automorphism.

Several authors calculated the Nakayama automorphism of skew (or twisted) Calabi-Yau algebras [BrZ, GY, LWW, RRZ1, RRZ2, Ye]. In general the Nakayama automorphism is a subtle invariant and is difficult to compute. Several researchers have been investigating the Nakayama automorphism and its applications. In [RRZ1, RRZ2] Rogalski-Reyes-Zhang proved several homological identities about the Nakayama automorphism. Liu-Wang-Wu studied the Nakayama automorphism for Ore extensions; in particular, they gave a description of the Nakayama automorphism of $A[x; \sigma, \delta]$ [LWW, Theorem 0.2].

This paper will focus on the following classes of AS regular algebras:

$A(1) = k_{p_{ij}}[t_1, t_2, t_3]$, where $p_{ij} \in k^\times := k \setminus \{0\}$, is generated by t_1, t_2, t_3 and subject to the relations

$$(E0.3.1) \quad t_j t_i = p_{ij} t_i t_j, \quad \text{for all } 1 \leq i < j \leq 3.$$

$A(2)$ is generated by t_1, t_2, t_3 and subject to the relations

$$(E0.3.2) \quad t_1 t_2 - t_2 t_1 = t_1 t_3 - t_3 t_1 = t_3 t_2 - p t_2 t_3 - t_1^2 = 0$$

where $p \in k^\times$.

$A(3)$ is generated by t_1, t_2, t_3 and subject to the relations

$$(E0.3.3) \quad (t_2 + t_1) t_1 - t_1 t_2 = t_3 t_1 - q t_1 t_3 = t_3 t_2 - q(t_2 + t_1) t_3 = 0$$

where $q \in k^\times$.

$A(4)$ is generated by t_1, t_2, t_3 and subject to the relations

$$(E0.3.4) \quad (t_2 + t_1) t_1 - t_1 t_2 = t_3 t_1 - p t_1 t_3 = t_3 t_2 - p t_2 t_3 = 0$$

where $p \in k^\times$.

$A(5)$ is generated by t_1, t_2, t_3 and subject to the relations

$$(E0.3.5) \quad (t_2 + t_1) t_1 - t_1 t_2 = (t_3 + t_2 + t_1) t_1 - t_1 t_3 = (t_3 + t_2 + t_1) t_2 - (t_2 + t_1) t_3 = 0.$$

$A(6)$ is the graded down-up algebra $A(\alpha, \beta)$ which is generated by x, y and subject to the relations

$$(E0.3.6) \quad x^2 y - \alpha x y x - \beta y x^2 = x y^2 - \alpha y x y - \beta y^2 x = 0$$

where $\alpha \in k$ and $\beta \in k^\times$.

$A(7) := S(p)$ is generated by x, y and subject to the relations

$$(E0.3.7) \quad x^2y - pyx^2 = xy^2 + py^2x = 0$$

where $p \in k^\times$.

These are some pair-wise non-isomorphic noetherian AS regular algebras of global dimension three. The Nakayama automorphism of these algebras will be given explicitly in Section 1.

Going back to noncommutative invariant theory, we adapt the standard hypotheses given in [CWZ]. The first application of the Nakayama automorphism is the following result which answers Questions 0.1 and 0.2(2) in some special cases.

Theorem 0.4. *Let A be one of the following algebras and H act on A satisfying Hypothesis 2.1. Then H is a commutative group algebra.*

- (1) $A(1)$ where elements $p_{12}^{-2}p_{23}p_{31}$, $p_{31}^{-2}p_{12}p_{23}$ and $p_{23}^{-2}p_{31}p_{12}$ are not roots of unity. A special case is when p_{12} and p_{13} are roots of unity and p_{23} is not.
- (2) $A(2)$ where p is not a root of unity.
- (3) $A(3)$ where q is not a root of unity.
- (4) $A(4)$ where p is not a root of unity.
- (5) $A(5)$.
- (6) $A(6) = A(\alpha, \beta)$ where $\alpha \neq 0$ and β is not a root of unity.

The commutative group algebra H in Theorem 0.4 can be described explicitly. Theorem 0.4 also holds for the following AS regular algebras.

- (0.4.8) The class **D** of AS regular algebras of global dimension five defined in [LWW] with generic parameters.
- (0.4.9) The class **G** of AS regular algebras of global dimension five defined in [LWW] with generic parameters.

Note that if $A = A(0, \beta)$ (or $S(p)$), then there are non-group actions on A [Proposition 2.17]. AS regular algebras of global dimension three were classified by Artin, Schelter, Tate and Van den Bergh [AS, ATV1, ATV2]. The algebras listed before Theorem 0.4 are only a subset in their classification, nevertheless, Theorem 0.4 answers Question 0.1 in some special cases. Note that all algebras in Theorem 0.4 are not PI. In the PI case, we have the following result when H is semisimple and the H -action has trivial homological determinant, which partially answers Question 0.2(4).

Theorem 0.5. *Assume that A is one of the following algebras. Let H be a semisimple Hopf algebra acting on A and satisfying Hypothesis 2.1. Suppose the H -action has trivial homological determinant. Then H is the dual of a group algebra.*

- (1) $A(1)$ where $p_{12}^{-2}p_{23}p_{31}$, $p_{31}^{-2}p_{12}p_{23}$ and $p_{23}^{-2}p_{31}p_{12}$ are not equal to 1.
- (2) $A(2)$ where $p \neq \pm 1$.
- (3) $A(3)$ (even without “trivial homological determinant” hypothesis).
- (4) $A(4)$ (even without “trivial homological determinant” hypothesis).
- (6) $A(6)$ where $\beta \neq \pm 1$.
- (7) $A(7)$ where $p \neq \pm i$.

In cases (1)-(4), H is also a commutative group algebra.

Theorem 0.5 also holds for the following algebras.

(0.5.8) The class **D** of AS regular algebras of global dimension five defined in [LWW] with $p^{-6}q^8 \neq 1$.

(0.5.9) The class **G** of AS regular algebras of global dimension five defined in [LWW] with $g \neq \pm 1$.

See Corollaries 2.6 and 2.7 for other related results.

The second application of the Nakayama automorphism relies on the following theorem.

Theorem 0.6. *Let A be a connected graded domain. Then its Nakayama automorphism (if exists) commutes with any algebra automorphism of A .*

A slightly weaker version of Theorem 0.6 was proved in [RRZ1, Theorem 3.11], which states that μ_A commutes with graded algebra automorphisms of A . As a consequence of Theorem 0.6 we have the following, which partially answers the second part of Question 0.2(5).

Corollary 0.7. *If A is one of the following algebras, then every algebra automorphism of A preserves the \mathbb{N} -grading of A .*

- (1) $A(1)$ where p_{ij} are generic or $(p_{12}, p_{13}, p_{23}) = (1, 1, p)$ with p not a root of unity.
- (2-4) $A(2)$ - $A(4)$ with the same hypotheses as in Theorem 0.4(2-4).
- (6) $A(6)$ where β is not a root of unity.
- (7) $A(7)$ where p is not a root of unity.

Using Corollary 0.7, the automorphism group of the algebras are calculated explicitly, see Section 4. Note that there are non-graded algebra automorphisms for algebra $A(5)$, see Theorem 5.8. A much more interesting and difficult question is the first part of Question 0.2(5). For example, we ask whether every finite dimensional Hopf action on the algebras in Corollary 0.7 preserve the \mathbb{N} -grading.

In the papers [CPWZ1, CPWZ2, BeZ] the authors use the discriminant to control algebra automorphisms and locally nilpotent derivations. The third application of the Nakayama automorphism concerns with locally nilpotent derivations and the cancellation problem.

Theorem 0.8. *Let A be a connected graded domain. Then its Nakayama automorphism (if exists) commutes with any locally nilpotent derivation of A .*

Using the similar ideas to those presented in [BeZ], we can solve Zariski Cancellation Problem for some noncommutative algebras. Recall that an algebra A is called *cancellative* if $A[t] \cong B[t]$ for any algebra B implies that $A \cong B$. The original Zariski Cancellation Problem asks if the commutative polynomial ring $k[t_1, \dots, t_n]$ is cancellative. Recall that $k[t_1]$ is cancellative by a result of Abhyankar-Eakin-Heinzer [AEH], $k[t_1, t_2]$ is cancellative by Fujita [Fu] and Miyanishi-Sugie [MS] in characteristic zero and by Russell [Ru] in positive characteristic. The Zariski Cancellation Problem was open for many years. In 2013, a remarkable development was made by Gupta [Gu1, Gu2] who completely settled this problem negatively in positive characteristic for $n \geq 3$. The Zariski Cancellation Problem in characteristic zero remains open for $n \geq 3$. Note that the Zariski Cancellation Problem is also related to Jacobian Conjecture. See [BeZ] for some background about cancellation problems. Here we solve the Zariski Cancellation Problem (ZCP) for some noncommutative algebras of dimension three.

Corollary 0.9. *Let A be any algebra in Corollary 0.7. Then A is cancellative.*

The paper is organized as follows. In Section 1, we recall the definition of the Nakayama automorphism and AS regular algebras and compute the Nakayama automorphism of the algebras in Theorem 0.4. The proofs of Theorem 0.4 and Theorem 0.5 are given in Sections 2 and 3. In Section 4, we prove Theorem 0.6 and Corollary 0.7. In Section 5, we calculate the full automorphism group of $A(5)$. In Section 6, we prove Theorem 0.8 and Corollary 0.9.

1. NAKAYAMA AUTOMORPHISM

Throughout let A be an algebra over k . Let A^e denote the enveloping algebra $A \otimes A^{op}$, where A^{op} is the opposite ring of A . An A -bimodule can be identified with a left A^e -module.

Definition 1.1. [RRZ1, Definition 0.1] Let A be an algebra over k .

- (a) A is called *skew Calabi-Yau* (or *skew CY*, for short) if
 - (i) A is homologically smooth, that is, A has a projective resolution in the category $A^e\text{-Mod}$ that has finite length and such that each term in the projective resolution is finitely generated, and
 - (ii) there is an integer d and an algebra automorphism μ of A such that

$$(E1.1.1) \quad \text{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} 0 & i \neq d \\ {}^1A^\mu & i = d, \end{cases}$$

as A -bimodules, where 1 denotes the identity map of A .

- (b) If (E1.1.1) holds for some algebra automorphism μ of A (even if A is not skew CY), then μ is called the *Nakayama automorphism* of A , and is usually denoted by μ_A .
- (c) [Gi, Definition 3.2.3] We call A *Calabi-Yau* (or *CY*, for short) if A is skew Calabi-Yau and μ_A is inner.

Definition 1.2. A connected graded algebra A is called *Artin-Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

- (a) A has finite injective dimension $d < \infty$ on both sides,
- (b) $\text{Ext}_A^i(k, A) = \text{Ext}_{A^{op}}^i(k, A) = 0$ for all $i \neq d$ where $k = A/A_{\geq 1}$, and
- (c) $\text{Ext}_A^d(k, A) \cong k(l)$ and $\text{Ext}_{A^{op}}^d(k, A) \cong k(l)$ for some integer l . The integer l is called the AS index.

If moreover

- (d) A has (graded) finite global dimension d , then A is called *Artin-Schelter regular* (or *AS regular*, for short).

By [RRZ1, Lemma 1.2], if A is connected graded, then A is AS regular if and only if A is skew CY.

We will use the following two lemmas several times.

Lemma 1.3. [RRZ1, Lemma 1.5] *Let A be a noetherian connected graded AS Gorenstein algebra and let z be a homogeneous normal regular element of positive degree such that $\mu_A(z) = cz$ for some $c \in k^\times$. Let τ be in $\text{Aut}(A)$ such that $za = \tau(a)z$ for all $a \in A$. Then $\mu_{A/(z)}$ is equal to $\mu_A \circ \tau$ when restricted to $A/(z)$.*

Given a nonzero scalar $c \in k^\times$, we define a graded algebra automorphism ξ_c of A by

$$\xi_c(a) = c^{\deg a} a$$

for all homogeneous elements $a \in A$.

Lemma 1.4. [RRZ1, Theorem 0.3] *Let A be a noetherian connected graded AS Gorenstein algebra with AS index l . Let A^σ be the graded twist of A associated to a graded automorphism σ of A . Then $\mu_{A^\sigma} = \mu_A \circ \sigma^l \circ \xi_{\text{hdet}(\sigma)}^{-1}$.*

Lemma 1.4 holds in the multi-graded case, see [RRZ1, Theorem 5.4(a)]. We now compute the Nakayama automorphism of some classes of AS regular algebras of dimension three (or higher).

First we consider the skew polynomial ring $A = k_{p_{ij}}[t_1, \dots, t_n]$ which is generated by t_1, \dots, t_n and subject to the relations

$$t_j t_i = p_{ij} t_i t_j$$

for all $i < j$, where $p_{ij} \in k^\times$ are nonzero scalars satisfying $p_{ii} = 1$ for all i and $p_{ji} = p_{ij}^{-1}$ for all i, j . The Nakayama automorphism of A is calculated in [LWW, Proposition 4.1] and [RRZ1, Example 5.5]:

$$\mu_A : t_i \mapsto \left(\prod_{s=1}^n p_{si} \right) t_i, \quad \text{for all } i.$$

If n is three, then the Nakayama automorphism of $k_{p_{ij}}[t_1, t_2, t_3]$ is determined by

$$(E1.5.1) \quad \mu_A : t_1 \rightarrow p_{21}p_{31} t_1, \quad t_2 \rightarrow p_{12}p_{32} t_2, \quad t_3 \rightarrow p_{13}p_{23} t_3.$$

The next algebra is

$$A = k\langle t_1, t_2, \dots, t_n \rangle / (t_n t_{n-1} - p t_{n-1} t_n - \sum_{i=1}^s t_i^2, t_i \text{ is central for all } i \leq n-2)$$

where $p \in k^\times$ and s is a positive integer no more than $n-2$. Let $\Omega = \sum_{i=1}^s t_i^2$. Then Ω is central and $\mu_A(\Omega) = \Omega$. By Lemma 1.3,

$$\mu_A = \mu_{A/(\Omega)} = \mu_{B/(\Omega)} = \mu_B$$

where

$$B = k\langle t_1, t_2, \dots, t_n \rangle / (t_n t_{n-1} - p t_{n-1} t_n, t_i \text{ is central for all } i \leq n-2).$$

Hence

$$(E1.5.2) \quad \mu_A : t_i \rightarrow t_i, \quad \forall i \leq n-2, \quad \text{and} \quad t_{n-1} \rightarrow p^{-1} t_{n-1}, \quad t_n \rightarrow p t_n$$

If $n = 3$ and $s = 1$, then

$$\mu_A : t_1 \rightarrow t_1, \quad t_2 \rightarrow p^{-1} t_2, \quad t_3 \rightarrow p t_3.$$

Consider the commutative polynomial ring $B = k[t_1, t_2, t_3]$ as a \mathbb{Z} -graded algebra with $\deg t_i = 1$ for all $i = 1, 2, 3$. Let σ be a graded algebra automorphism of B determined by $t_1 \rightarrow t_1, t_2 \rightarrow t_2 - t_1, t_3 \rightarrow q t_3$ where q is a nonzero scalar. We use the convention in [RRZ1, Section 5] to deal with the graded twist, so we can use the identity proved in [RRZ1, Theorem 5.4(a)]. The new multiplication of the graded twist, denoted by B^σ , associated to σ is defined by [RRZ1, (E5.0.2)]. Starting from

a commutative relation $t_j t_i = t_i t_j$, we have a relation $\sigma^{-1}(t_j) t_i = \sigma^{-1}(t_i) t_j$ in the graded twist B^σ . Hence the relations for the algebra B^σ are

$$(t_2 + t_1)t_1 = t_1 t_2, \quad q^{-1} t_3 t_1 = t_1 t_3, \quad q^{-1} t_3 t_2 = (t_2 + t_1)t_3$$

which show that B^σ is isomorphic to the $A(3)$ given before Theorem 0.4. Note that $l = 3$ for the algebra B and $\text{hdet } \sigma = q$, using the identity

$$\mu_{B^\sigma} = \mu_B \circ \sigma^l \circ \xi_{\text{hdet } \sigma}^{-1}$$

[Lemma 1.4] or [RRZ1, Theorem 5.4(a)], we have a formula for the Nakayama automorphism of $A(3)$:

$$(E1.5.3) \quad \mu_{A(3)} : t_1 \rightarrow q^{-1} t_1, \quad t_2 \rightarrow q^{-1}(t_2 - 3t_1), \quad t_3 \rightarrow q^2 t_3.$$

For the next algebra, we need to work with a \mathbb{Z}^2 -graded twist. Let B be the commutative polynomial ring $k[t_1, t_2, t_3]$ with \mathbb{Z}^2 -grading determined by $\deg t_1 = \deg t_2 = (1, 0)$ and $\deg t_3 = (0, 1)$. Consider two graded algebra automorphisms $\sigma_1 : t_1 \rightarrow t_1, t_2 \rightarrow t_2 - t_1, t_3 \rightarrow t_3$ and $\sigma_2 : t_1 \rightarrow p^{-1} t_1, t_2 \rightarrow p^{-1} t_2, t_3 \rightarrow t_3$ where p is a nonzero scalar. Define a twisting system $\sigma = \{\sigma_{a,b} = \sigma_1^a \sigma_2^b \mid (a, b) \in \mathbb{Z}^2\}$. The graded twist B^σ is isomorphic to

$$A(4) = k\langle t_1, t_2, t_3 \rangle / ((t_2 + t_1)t_1 - t_1 t_2, t_3 t_1 - p t_1 t_3, t_3 t_2 - p t_2 t_3).$$

Note that $l = (2, 1)$ and $\text{hdet } \sigma = (1, p^{-2})$. By [RRZ1, Theorem 5.4(a)], we have a formula for the Nakayama automorphism

$$(E1.5.4) \quad \mu_{A(4)} : t_1 \rightarrow p^{-1} t_1, \quad t_2 \rightarrow p^{-1}(t_2 - 2t_1), \quad t_3 \rightarrow p^2 t_3.$$

The algebra

$$A(5) := k\langle t_1, t_2, t_3 \rangle / ((t_2 + t_1)t_1 - t_1 t_2, (t_3 + t_2 + t_1)t_1 - t_1 t_3, (t_3 + t_2 + t_1)t_2 - (t_2 + t_1)t_3)$$

is a graded twist B^σ with $\sigma : t_1 \rightarrow t_1, t_2 \rightarrow t_2 - t_1, t_3 \rightarrow t_3 - t_2$. So the Nakayama automorphism of $A(5)$ is determined by

$$(E1.5.5) \quad \mu_{A(5)} : t_1 \rightarrow t_1, \quad t_2 \rightarrow t_2 - 3t_1, \quad t_3 \rightarrow t_3 - 3t_2 + 3t_1.$$

Graded down-up algebras $A(\alpha, \beta)$ have been studied by several researchers. By definition, $A(\alpha, \beta)$ is generated by x and y and subject to the relations

$$x^2 y - \alpha x y x - \beta y x^2, \quad x y^2 - \alpha y x y - \beta y^2 x.$$

This is an AS regular algebra when $\beta \neq 0$. This class of algebras are not Koszul, but 3-Koszul. Graded automorphisms of $A(\alpha, \beta)$ have been worked out in [KK]. Consider the characteristic equation

$$w^2 - \alpha w - \beta = 0$$

and let w_1 and w_2 be the roots of the above equation. Then $\Omega := xy - w_1 yx$ is a normal regular element and $A(\alpha, \beta)/(\Omega)$ is a skew polynomial ring of global dimension two. Since we know the Nakayama automorphism of $A(\alpha, \beta)/(\Omega)$ by [LWW, Proposition 4.1], Lemma 1.3 shows that the Nakayama automorphism of $A(6) := A(\alpha, \beta)$ is determined by

$$(E1.5.6) \quad \mu_{A(6)} : x \rightarrow -\beta x, \quad y \rightarrow -\beta^{-1} y.$$

Another class of non-Koszul AS regular is $A(7) := S(p)$ which is generated by x and y and subject to relations

$$x^2 y - p y x^2, \quad x y^2 + p y^2 x$$

where $p \in k^\times$. Note that $z := y^2$ is a normal element such that $\mu_{A(7)}(z) = cz$ since the Nakayama automorphism preserves \mathbb{Z}^2 -grading of $A(7)$. It is easy to see that $za = \tau(a)z$ where $\tau \in \text{Aut}(A(7))$ maps x to $-p^{-1}x$ and y to y . By Lemma 1.3, $\mu_{A(7)/(z)} = \mu_{A(7)} \circ \tau$. Applying Lemma 1.3 to the algebra $A(6)$ with $z = y^2$, we have $\mu_{A(6)/(z)} = \mu_{A(6)} \circ \tau'$ where τ' maps x to $\beta^{-1}x$ and y to y . Note that $A(6)/(y^2)|_{\alpha=0, \beta=p} = A(7)/(y^2)$. Therefore, when $\alpha = 0$ and $\beta = p$,

$$\mu_{A(7)} \circ \tau = \mu_{A(7)/(z)} = \mu_{A(6)/(z)} = \mu_{A(6)} \circ \tau'.$$

Then, by an easy calculation and (E1.5.6), we obtain that

$$(E1.5.7) \quad \mu_{A(7)} : x \rightarrow px, \quad y \rightarrow -p^{-1}y.$$

Finally, let us mention two classes of AS regular algebras of dimension five for which the Nakayama automorphism has been computed by Liu-Wang-Wu [LWW]. As in [LWW], the algebras \mathbf{D} and \mathbf{G} are of the form $k\langle x, y \rangle(r_1, r_2, r_3)$ where r_i are relations. For the algebra \mathbf{D} , the three relations are

$$\begin{aligned} r_{D1} &= x^3y + px^2yx + qxyx^2 - p(2p^2 + q)yx^3, \\ r_{D2} &= x^2y^2 - p(p^2 + q)yxyx - q^2y^2x^2 + (q - p^2)xy^2x + (q - p^2)yx^2y, \\ r_{D3} &= xy^3 + pyxy^2 + qy^2xy - p(2p^2 + q)y^3x, \end{aligned}$$

where $p, q \in k^\times$ and $2p^4 - p^2q + q^2 = 0$. By [LWW, Theorem 4.3(1)], the Nakayama automorphism of \mathbf{D} is given by

$$(E1.5.8) \quad \mu_{\mathbf{D}} : x \rightarrow p^{-3}q^4x, \quad y \rightarrow p^3q^{-4}y.$$

For the algebra \mathbf{G} , the three relations are

$$\begin{aligned} r_{G1} &= x^3y + px^2yx + qxyx^2 + syx^3, \\ r_{G2} &= x^2y^2 + l_2xyxy + l_3yxyx + l_4y^2x^2 + l_5xy^2x + l_5yx^2y, \\ r_{G3} &= xy^3 + pyxy^2 + qy^2xy + sy^3x, \end{aligned}$$

where

$$l_2 = \frac{-s^2(qs - g)}{g(qs + g)}, \quad l_3 = s - \frac{pg(ps - q^2)}{q(qs + g)}, \quad l_4 = \frac{-g^2}{s^2}, \quad l_5 = \frac{ps^2 + qg}{qs + g},$$

with $p, q, s, g \in k^\times$, $ps^3g + qsg^2 + s^5 + g^3 = 0$, $p^3s = q^3$, $ps \neq q^2$, $q^2s^2 \neq g^2$ and $s^5 + g^3 \neq 0$. By [LWW, Theorem 4.3(2)], the Nakayama automorphism of \mathbf{G} is given by

$$(E1.5.9) \quad \mu_{\mathbf{G}} : x \rightarrow gx, \quad y \rightarrow g^{-1}y.$$

The above are all AS regular algebras that we will be dealing with.

2. HOPF ACTIONS ON A WITH DIAGONALIZABLE μ_A

In this and the next sections we study finite dimensional Hopf actions on AS regular algebras and partially answer Questions 0.1 and 0.2(2,4). We impose the following standard hypotheses for the rest of the section.

Hypothesis 2.1. [CWZ, Hypothesis 0.3] *We assume that*

- (i) H is a finite dimensional Hopf algebra.
- (ii) A is a connected graded AS regular algebra generated in degree 1.
- (iii) H acts on A inner faithfully, namely, there is no nonzero Hopf ideal $I \subset H$ such that $IA = 0$.

(iv) *The H -action on A preserves the grading of A .*

Let A be an AS regular algebra generated in degree 1 and let $V := A_1$ be the degree 1 piece of A . Let $K := H^\circ$ be the dual Hopf algebra of H . Then a left H -action on A is equivalent to a right K -coaction on A .

We say that a right K -coaction on A is inner faithful if for any proper Hopf subalgebra $K' \subsetneq K$, we have that $\rho(A) \not\subseteq A \otimes K'$. The main tool of this paper is the following homological identity proved in [CWZ, Theorem 0.1] with an improved version given in [RRZ2].

Theorem 2.2. [RRZ2, Theorem 4.3] *Let A be a noetherian AS regular algebra with Nakayama automorphism μ_A . Let K be a Hopf algebra with bijective antipode S coacting on A inner faithfully from the right. Suppose that the homological codeeterminant [CWZ, Definition 1.5(b)] of the K -coaction on A is the element $D \in K$. Then*

$$(E2.2.1) \quad \eta_D \circ S^2 = \eta_{\mu_A^\tau}$$

where η_D is the automorphism of K defined by conjugating by D and $\eta_{\mu_A^\tau}$ is the automorphism of K given by conjugating by the transpose of the corresponding matrix of μ_A .

Here are some explanations from [CWZ]. The automorphism on the left-hand side of equation (E2.2.1) is the composition of the Hopf algebra automorphism S^2 of K (which is bijective by hypothesis) and the Hopf algebra automorphism η_D of K where η_D is given by $\eta_D(a) = D^{-1}aD$ for all $a \in K$. To understand the right-hand side of equation (E2.2.1) we start with a k -linear basis, say $\{v_1, \dots, v_n\}$ of A_1 , the degree 1 graded piece of A . Then the Nakayama automorphism μ_A can be written as

$$(E2.2.2) \quad \mu_A(v_i) = \sum_{j=1}^n m_{ij} v_j, \quad \text{for all } i = 1, \dots, n.$$

Let \mathbb{M} be the $n \times n$ -matrix $(m_{ij})_{n \times n}$ over the base field k . Let $\rho : A \rightarrow A \otimes K$ be the right K -coaction on A . Then

$$(E2.2.3) \quad \rho(v_i) = \sum_j v_j \otimes y_{ji}, \quad \text{for all } i = 1, \dots, n$$

for some $y_{ji} \in K$. Then $\Delta(y_{st}) = \sum_{j=1}^n y_{sj} \otimes y_{jt}$ and $\epsilon(y_{st}) = \delta_{st}$ for all s, t . Let ρ^* be the left K -coaction on the Ext-algebra $E := \text{Ext}_A^*(k, k)$ induced by the K -coaction on A . Then we have

$$\rho^*(v_i^*) = \sum_{s=1}^n y_{is} \otimes v_s^*, \quad \text{for all } i = 1, \dots, n.$$

Since the K -coaction on A is inner faithful, $\{y_{ij}\}_{1 \leq i, j \leq n}$ generates K as a Hopf algebra. With this choice of basis $\{v_i\}_{i=1}^n$, we define

$$(E2.2.4) \quad \eta_{\mu_A^\tau} : y_{ij} \rightarrow \sum_{s,t=1}^n m_{si} y_{st} w_{jt}$$

for all $1 \leq i, j \leq n$ where the matrix $\mathbb{W} := (w_{ij})_{n \times n} = (m_{ij})_{n \times n}^{-1}$. Roughly speaking, we use coordinates to define the conjugation automorphism $\eta_{\mu_A^\tau}$ of K . Theorem

2.2 implies that the definition of this automorphism is independent of the choice of coordinates.

Since K is finite dimensional, $\eta_D \circ S^2$ has finite order. So it follows from (E2.2.1) that $\eta_{\mu_A^\tau}$ is of finite order. In fact, the order of $\eta_{\mu_A^\tau}$ divides $2 \dim_k K$, see the proof of [CWZ, Theorem 4.3].

In this section, we consider the case when \mathbb{M} is a diagonal matrix. Let A be a noetherian AS regular algebra generated by $A_1 = V$. We say the Nakayama automorphism μ_A has eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ if there is a k -linear basis $\{v_1, \dots, v_n\}$ of V such that $\mu(v_i) = \lambda_i v_i$ for all $i = 1, \dots, n$. From now on we assume Hypothesis 2.1.

Lemma 2.3. *Let A be an AS regular algebra generated by $V = A_1$ and μ_A be the Nakayama automorphism of A . Suppose μ_A has eigenvalues $\{\lambda_1, \dots, \lambda_n\}$.*

- (1) *If $\lambda_i \lambda_j^{-1}$ is not a root of unity for a pair (i, j) , then $y_{ij} = y_{ji} = 0$.*
- (2) *If $\lambda_i \lambda_j^{-1}$ is not a root of unity for all $i \neq j$, then K is a group algebra. As a consequence, H and K are semisimple.*
- (3) *Suppose the setting of part (2). If, further, $v_j v_i = p_{ij} v_i v_j \neq 0$ for all $i \neq j$ for some $p_{ij} \in k^\times$, then H and K are commutative group algebras.*

For the next three parts, we further assume that H is semisimple and the H -action on A has trivial homological determinant.

- (4) *If $\lambda_i \lambda_j^{-1}$ is not 1 for a pair (i, j) , then $y_{ij} = y_{ji} = 0$.*
- (5) *If $\lambda_i \lambda_j^{-1}$ is not 1 for all $i \neq j$, then K is a group algebra. As a consequence, H and K are semisimple.*
- (6) *Suppose the setting of part (5). If, further, $v_j v_i = p_{ij} v_i v_j \neq 0$ for all $i \neq j$ for some $p_{ij} \in k^\times$, then H and K are commutative group algebras.*

Proof. (1) By (E2.2.4),

$$\eta_{\mu_A^\tau}(y_{ij}) = \lambda_i \lambda_j^{-1} y_{ij}.$$

By Theorem 2.2, $\eta_D \circ S^2(y_{ij}) = \eta_{\mu_A^\tau}(y_{ij})$ for all i, j . Since K is finite dimensional, the automorphism $\eta_D \circ S^2$ has finite order. Hence, $\eta_{\mu_A^\tau}$ has finite order, and therefore there is an N such that

$$y_{ij} = (\eta_{\mu_A^\tau})^N(y_{ij}) = (\lambda_i \lambda_j^{-1})^N y_{ij}.$$

Since $\lambda_i \lambda_j^{-1}$ is not a root of unity, $y_{ij} = 0$. By symmetry, $y_{ji} = 0$.

(2) If $\lambda_i \lambda_j^{-1}$ is not a root of unity for all $i \neq j$, then $y_{ij} = 0$ for all $i \neq j$ by part (1). Thus $\rho(v_i) = v_i \otimes y_{ii}$ and y_{ii} is a group-like element. Since the K -coaction on A is inner faithful, K is generated by $\{y_{ii}\}_{i=1}^n$. So K is a group algebra.

(3) Applying ρ to the equation $v_j v_i = p_{ij} v_i v_j$, we obtain that

$$v_j v_i \otimes y_{jj} y_{ii} = \rho(v_j v_i) = p_{ij} \rho(v_i v_j) = p_{ij} v_i v_j \otimes y_{ii} y_{jj} = v_j v_i \otimes y_{ii} y_{jj}.$$

Hence y_{ii} commutes with y_{jj} . So K is a commutative group algebra. By [Mo, Theorem 2.3.1], H is a commutative group algebra.

(4-6) We have $S^2 = Id$ and $D = 1_K$ since H (and hence K) is semisimple and the H -action has trivial homological determinant. By Theorem 2.2, $\eta_{\mu_A^\tau}$ is the identity map of K . Recall that $\eta_{\mu_A^\tau}(y_{ij}) = \lambda_i \lambda_j^{-1} y_{ij}$. If $\lambda_i \lambda_j \neq 1$, then $y_{ij} = 0$. The rest of the argument is similar to the proof of (1-3). \square

Proposition 2.4. *Let A and B be AS regular algebras generated in degree 1 such that $A \otimes B$ is noetherian. Assume that the Nakayama automorphisms μ_A and μ_B have eigenvalues $\{\lambda_1, \dots, \lambda_m\}$ and $\{\lambda_{m+1}, \dots, \lambda_{m+n}\}$ respectively. Suppose that*

- (1) *The only Hopf (resp. semisimple Hopf) actions on A and B are group actions.*
- (2) *$\lambda_i \lambda_j^{-1}$ are not roots of unity for all $i \leq m$ and $j > m$.*

Then every Hopf (resp. semisimple Hopf) action on $A \otimes B$ is a group action.

Proof. Let H act on $A \otimes B$ satisfying Hypothesis 2.1. Then $K := H^\circ$ coacts on A . Since $\mu_{A \otimes B} = \mu_A \otimes \mu_B$, there is a basis $\{v_1, \dots, v_m\} \cup \{v_{m+1}, \dots, v_{m+n}\}$ such that $\mu_{A \otimes B}$ has eigenvalues $\{\lambda_1, \dots, \lambda_{m+n}\}$ with respect to this basis, where $A_1 = \sum_{i=1}^m kv_i$ and $B_1 = \sum_{i=m+1}^{m+n} kv_i$. By Lemma 2.3(1), $y_{ij} = 0$ if $(i \leq m, j > m)$ or $(i > m, j \leq m)$.

We make the following remark for the semisimple case. If H is semisimple, by Larson-Radford [LR1, LR2], K is semisimple, cosemisimple and involutory (namely, $S^2 = Id$). So every Hopf subalgebra of K is involutory, whence semisimple and cosemisimple.

Let K_A be the Hopf subalgebra of K generated by $\{y_{ij} \mid 1 \leq i, j \leq m\}$ and K_B be the Hopf subalgebra of K generated by $\{y_{ij} \mid m+1 \leq i, j \leq m+n\}$. So K_A coacts on A inner faithfully. Now the dual algebra $(K_A)^\circ$ acts on A satisfying Hypothesis 2.1. (when H is semisimple, then so is $(K_A)^\circ$.) By hypothesis (1), $(K_A)^\circ$ is a group algebra, so K_A is commutative. Similarly, K_B is commutative. Any $i \leq m$ and $s \geq m+1$, v_i commutes with v_s . So, after applying ρ , we have that y_{ij} commutes with y_{st} for all $1 \leq i, j \leq m$ and $m+1 \leq s, t \leq m+n$.

Finally we will show that K is commutative. Since K is generated by K_A and K_B , it suffices to show that K_A commutes with K_B . Let $R = M_{n \times n}(K)$ and consider K as the diagonal subalgebra of R . For any subalgebra $B \subset R$, let $C_B(K)$ be the centralizer $\{X \in R \mid Xa = aX, \forall a \in B\}$. Let B be the subalgebra generated by $\{y_{st} \mid m+1 \leq s, t \leq m+n\}$. Let $X = (y_{ij})_{n \times n}$. Then $X \in C_B(K)$. By the antipode axiom, $X^{-1} = (S(y_{ij}))$. Since $X^{-1} \in C_B(K)$, $S(y_{ij})$ commutes with y_{st} for all $m+1 \leq s, t \leq m+n$. This K_A commutes with B . Similarly, one shows that K_B commutes with K_A . Now we have that K is commutative. Since the base field k is algebraically closed of characteristic zero, $H = K^\circ$ is a group algebra. \square

Remark 2.5. Proposition 2.4 applies to the “twisted” tensor product $A \otimes_q B$ under suitable hypotheses.

One immediate consequence is

Corollary 2.6. *Suppose $\{p_{ij} \mid 1 \leq i < j \leq m\}$ are generic parameters. Then every semisimple Hopf action on $k_{p_{ij}}[x_1, \dots, x_m] \otimes k[y_1, \dots, y_n]$ is a group action.*

Proof. Let $A = k_{p_{ij}}[x_1, \dots, x_m]$ and $B = k[y_1, \dots, y_n]$. By [CWZ, Theorem 4.3], A satisfies the hypothesis in Proposition 2.4(1). By [EW, Theorem 1.3], B satisfies the hypothesis in Proposition 2.4(1) (in the semisimple Hopf case). By [RRZ1, Example 5.5], the hypothesis in Proposition 2.4(2) holds. So the assertion follows. \square

Sometimes a similar idea applies to a non-tensor product. Here is an example.

Corollary 2.7. *Let $p \in k^\times$ be not a root of unity and $s \leq n-2$. Let*

$$A = k\langle t_1, t_2, \dots, t_n \rangle / (t_n t_{n-1} - p t_{n-1} t_n - (\sum_{i=1}^s t_i^2), t_i \text{ central for all } i \leq n-2).$$

Then every semisimple Hopf action on A is a group action.

Proof. The Nakayama automorphism of A is given in (E1.5.2)

$$\mu_A : t_{n-1} \rightarrow p^{-1}t_{n-1}, \quad t_n \rightarrow pt_n, \quad t_i \rightarrow t_i$$

for all $i \leq n-2$. So μ_A has eigenvalues $\{1, \dots, 1, p, p^{-1}\}$. So $\lambda_i \lambda_j^{-1}$ is not a root of unity when $i \neq j$ and at least one of (i, j) is either n or $n-1$. By Lemma 2.3(1), $y_{ij} = 0$ when $i \neq j$ and at least one of (i, j) is either n or $n-1$.

Let H be a semisimple Hopf algebra acting on A and $K = H^\circ$. So K and its Hopf subalgebras are semisimple.

Let K_1 be the Hopf subalgebra of K generated by y_{ij} for all $1 \leq i, j \leq n-2$. Thus K_1 coacts on $k[t_1, \dots, t_{n-2}]$. By [EW, Theorem, 1.3], K_1 is commutative. Also y_{nn} and y_{n-1n-1} are grouplike elements. Since t_n commutes with t_i for all $i \leq n-2$, y_{nn} commutes with K_1 . Similarly, y_{n-1n-1} commutes with K_1 . Applying ρ to the relation $t_n t_{n-1} - pt_{n-1} t_n - (\sum_{i=1}^s t_i^2)$, one sees that y_{nn} commutes with y_{n-1n-1} . Since K is generated by K_1 , y_{nn} and y_{n-1n-1} , K is commutative. Its dual Hopf algebra H is a group algebra. \square

Next we prove a part of Theorems 0.4 and 0.5. We assume Hypothesis 2.1.

Proposition 2.8. *Let A be the algebra $A(1)$, namely, the skew polynomial ring $k_{p_{ij}}[t_1, t_2, t_3]$, see (E0.3.1). Let H act on A .*

- (1) *If $p_{12}^{-2} p_{23} p_{31}$, $p_{31}^{-2} p_{12} p_{23}$ and $p_{23}^{-2} p_{31} p_{12}$ are not roots of unity, then H is a commutative group algebra.*
- (2) *Suppose that H is semisimple and the H -action has trivial homological determinant. If $p_{12}^{-2} p_{23} p_{31}$, $p_{31}^{-2} p_{12} p_{23}$ and $p_{23}^{-2} p_{31} p_{12}$ are not 1, then H is a commutative group algebra.*

Proof. By (E1.5.1), the Nakayama automorphism of A is

$$\mu_A : t_1 \rightarrow \lambda_1 t_1, t_2 \rightarrow \lambda_2 t_2, t_3 \rightarrow \lambda_3 t_3$$

where $\lambda_1 = p_{21} p_{31}$, $\lambda_2 = p_{12} p_{32}$ and $\lambda_3 = p_{13} p_{23}$. So $\lambda_1 \lambda_2^{-1} = p_{12}^{-2} p_{23} p_{31}$, $\lambda_1^{-1} \lambda_3 = p_{31}^{-2} p_{12} p_{23}$ and $\lambda_2 \lambda_3^{-1} = p_{23}^{-2} p_{31} p_{12}$.

(1) Under the hypothesis of (1), $\lambda_i \lambda_j^{-1}$ are not roots of unity for all $i \neq j$. The assertion follows from Lemma 2.3(3).

(2) Under the hypothesis of (2), $\lambda_i \lambda_j^{-1}$ are not 1 for all $i \neq j$. The assertion follows from Lemma 2.3(6). \square

A special case of Proposition 2.8 is when $(p_{12}, p_{13}, p_{23}) = (1, 1, p)$. If p is not a root of unity, then the hypothesis about p_{ij} in Proposition 2.8(1) holds. So any Hopf action is a group action. If $p \neq \pm 1, \pm i$, then the hypothesis about p_{ij} in Proposition 2.8(2) holds. So any Hopf action is a group action (assuming H is semisimple and H -action has trivial homological determinant).

Proposition 2.9. *Let A be the algebra $A(2)$, see (E0.3.2). Let H act on A .*

- (1) *If p is not a root of unity, then H is a commutative group algebra.*
- (2) *Suppose H is semisimple and the H -action has trivial homological determinant. If $p \neq \pm 1$, then H is a commutative group algebra.*

Proof. By (E1.5.2) (taking $n = 3$), the Nakayama automorphism of A is

$$\mu_A : t_1 \rightarrow \lambda_1 t_1, t_2 \rightarrow \lambda_2 t_2, t_3 \rightarrow \lambda_3 t_3$$

where $\lambda_1 = 1$, $\lambda_2 = p^{-1}$ and $\lambda_3 = p$. So $\lambda_1\lambda_2^{-1} = p$, $\lambda_1^{-1}\lambda_3 = p$ and $\lambda_2\lambda_3^{-1} = p^{-2}$.

(1) Under the hypothesis of (1), $\lambda_i\lambda_j^{-1}$ are not roots of unity for all $i \neq j$. The assertion follows from Lemma 2.3(3).

(2) Under the hypothesis of (2), $\lambda_i\lambda_j^{-1}$ are not 1 for all $i \neq j$. The assertion follows from Lemma 2.3(6). \square

The next example shows that if p is a root of unity, there are Hopf algebra (and non-semisimple Hopf algebra) actions.

Example 2.10. There are many (non-group) Hopf actions on $A(2)$ when p is a root of unity.

Let T be any finite dimensional Hopf algebra that acts on the skew polynomial ring $k_p[x_1, x_2]$ satisfying Hypothesis 2.1 and with trivial homological determinant. Such T -actions are classified in [CKWZ]. In particular, there are many semisimple Hopf algebras T which are not group algebras if $p = -1$ and there are many non-semisimple Hopf algebras T which are not group algebras if $p \neq \pm 1$.

Such an T acts on $A(2)$. Suppose

$$\begin{aligned} h \cdot x_1 &= f_{11}(h)x_1 + f_{12}(h)x_2, \\ h \cdot x_2 &= f_{21}(h)x_1 + f_{22}(h)x_2 \end{aligned}$$

for all $h \in T$ with some k -linear maps $f_{ij} : T \rightarrow k$. Then we define

$$\begin{aligned} h \cdot t_1 &= \epsilon(h)t_1, \\ h \cdot t_2 &= f_{11}(h)t_2 + f_{12}(h)t_3, \\ h \cdot t_3 &= f_{21}(h)t_2 + f_{22}(h)t_3 \end{aligned}$$

for all $h \in T$. It is easy to check that this is a well defined T -action on $A(2)$ satisfying Hypothesis 2.1 with trivial homological determinant.

Here is a general result dealing with the AS regular algebras generated by two elements.

Proposition 2.11. *Let A be a noetherian AS regular algebra generated by two elements in degree one, say by x and y . Suppose that μ_A maps x to $\lambda_1 x$ and y to $\lambda_2 y$ such that $\lambda_1^{-1}\lambda_2$ is not a root of unity. Assume further that A has a relation of the form*

$$r := w_1 m_1 x y m_2 + w_2 m_1 y x m_2 + w_3 z + \cdots = 0$$

where $w_1, w_2, w_3 \in k^\times$ and m_1, m_2, z are monomials. Assume that $\{m_1 y x m_2, z, \cdots\}$ are k -linearly independent in A . Then any Hopf algebra H which acts on A is a commutative group algebra.

Proof. We may assume that $w_1 = -1$. Suppose $\rho : A \rightarrow A \otimes K$ is the correspondence coaction. By Lemma 2.3(2), K is a group algebra and $\rho(x) = x \otimes y_{11}$ and $\rho(y) = y \otimes y_{22}$ for some group-like elements y_{11} and y_{22} in K . Write $m_1 = x^{a_1} y^{a_2} \cdots y^{a_s}$, $m_2 = x^{b_1} y^{b_2} \cdots y^{b_t}$ and $z = x^{c_1} y^{c_2} \cdots y^{c_u}$ for the monomials appearing in r . Applying ρ

to $0 = r$, we have

$$\begin{aligned}
0 &= \rho(r) = \rho(-m_1 x y m_2 + w_2 m_1 y x m_2 + w_3 z \cdots) \\
&= -m_1 x y m_2 \otimes (y_{11}^{a_1} y_{22}^{a_2} \cdots y_{22}^{a_s} y_{11} y_{22} y_{11}^{b_1} y_{22}^{b_2} \cdots y_{22}^{b_t}) \\
&\quad + w_2 m_1 y x m_2 \otimes (y_{11}^{a_1} y_{22}^{a_2} \cdots y_{22}^{a_s} y_{22} y_{11} y_{11}^{b_1} y_{22}^{b_2} \cdots y_{22}^{b_t}) + \cdots \\
&= (w_2 m_1 y x m_2 + \cdots) \otimes (y_{11}^{a_1} y_{22}^{a_2} \cdots y_{22}^{a_s} y_{11} y_{22} y_{11}^{b_1} y_{22}^{b_2} \cdots y_{22}^{b_t}) \\
&\quad + w_2 m_1 y x m_2 \otimes (y_{11}^{a_1} y_{22}^{a_2} \cdots y_{22}^{a_s} y_{22} y_{11} y_{11}^{b_1} y_{22}^{b_2} \cdots y_{22}^{b_t}) + \cdots \\
&= w_2 m_1 y x m_2 \otimes (y_{11}^{a_1} y_{22}^{a_2} \cdots y_{22}^{a_s} (y_{22} y_{11} - y_{11} y_{22}) y_{11}^{b_1} y_{22}^{b_2} \cdots y_{22}^{b_t}) + \cdots
\end{aligned}$$

which implies that $y_{11} y_{22} = y_{22} y_{11}$. So K is commutative and cocommutative. Since we assume that k is algebraically closed of characteristic zero, K (and then H) is a commutative group algebra. \square

Remark 2.12. By the proof of Proposition 2.11, the hypothesis of “ $\lambda_i \lambda_j^{-1}$ not being a root of unity” can be replaced by the hypothesis “that K is a group algebra with $\rho(x) = x \otimes y_{11}$ and $\rho(y) = y \otimes y_{22}$ ”.

Proposition 2.13. *Let A be the down-up algebra algebra $A(6) = A(\alpha, \beta)$, see (E0.3.6). Let H act on A and $K = H^\circ$.*

- (1) *If β is not a root of unity, then K is a group algebra.*
- (2) *If $\alpha \neq 0$ and β is not a root of unity, then H is a commutative group algebra.*
- (3) *Suppose H is semisimple and the H -action has trivial homological determinant. If $\beta \neq \pm 1$, then K is a group algebra. If further $\alpha \neq 0$, then H is a commutative group algebra.*

Proof. By (E1.5.6), the Nakayama automorphism of A is

$$\mu_A : x \rightarrow -\beta x, y \rightarrow -\beta^{-1} y$$

So $\lambda_1 \lambda_2^{-1} = \beta^2$.

(1,2) Under the hypothesis of (1), $\lambda_1 \lambda_2^{-1}$ is not a root of unity. By Lemma 2.3(2), K is a group algebra. So part (1) follows. In part (2), we further assume that $\alpha \neq 0$, then Proposition 2.11 applies. So H is a commutative group algebra.

(3) Under the hypothesis of (3), $\lambda_1 \lambda_2^{-1}$ is not 1. By Lemma 2.3(5), K is a group algebra. When $\alpha \neq 0$, the assertion follows from Remark 2.12. \square

We will state a few more results. The proofs are very similar to the proof of Proposition 2.13, so we decide to omit them.

Proposition 2.14. *Let A be the algebra $A(7) = S(p)$, see (E0.3.7). Let H act on A and $K = H^\circ$.*

- (1) *If p is not a root of unity, then K is a group algebra.*
- (2) *Suppose H is semisimple and the H -action has trivial homological determinant. If $p \neq \pm i$, then K is a group algebra.*

Proposition 2.15. *Let A be the algebra \mathbf{D} with parameter (p, q) , see the end of Section 1. Let H act on A .*

- (1) *If $p^{-3} q^4$ is not a root of unity, then H is a commutative group algebra.*
- (2) *Suppose H is semisimple and the H -action has trivial homological determinant. If $p^{-3} q^4 \neq \pm 1$, then H is a commutative group algebra.*

Proposition 2.16. *Let A be the algebra \mathbf{G} , see the end of Section 1. Let H act on A .*

- (1) *If g is not a root of unity, then H is a commutative group algebra.*
- (2) *Suppose H is semisimple and the H -action has trivial homological determinant. If $q \neq \pm 1$, then H is a commutative group algebra.*

It is expected that the above theorem holds for the class of AS regular algebras of global dimension four with two generators and generic parameters.

Finally we classify all possible finite dimensional Hopf algebra actions on $A(0, \beta)$ and $S(p)$ that satisfies Hypothesis 2.1.

Proposition 2.17. *Let A be $A(0, \beta)$ or $S(p)$ where β and p are not roots of unity. Consider the Hopf algebra actions on A satisfying Hypothesis 2.1. Then H acts on A if and only if the dual Hopf algebra $K = H^\circ$ is a group algebra kG where G is a quotient group of $\langle a, b \mid a^2b = ba^2, ab^2 = b^2a \rangle$.*

Proof. If H acts on A , then we have a right K -coaction $\rho : A \rightarrow A \otimes K$ with $\rho(x) = x \otimes a_{11} + y \otimes a_{21}$ and $\rho(y) = x \otimes a_{12} + y \otimes a_{22}$, for some $a_{11}, a_{12}, a_{21}, a_{22} \in K$. By E1.5.6 and E1.5.7, we have

$$\mu_A : \begin{cases} x \rightarrow -\beta x, & y \rightarrow -\beta^{-1}y, & \text{if } A = A(0, \beta) \\ x \rightarrow px, & y \rightarrow -p^{-1}y, & \text{if } A = S(p). \end{cases}$$

Hence $\lambda_1 \lambda_2^{-1} = \begin{cases} \beta^2, & \text{if } A = A(0, \beta) \\ -p^2, & \text{if } A = S(p) \end{cases}$ is not a root of unity. By Lemma 2.3,

$a_{12} = a_{21} = 0$. Thus $\rho(x) = x \otimes a_{11}$ and $\rho(y) = y \otimes a_{22}$. Since the K -coaction on A is inner faithful, $\{a_{11}, a_{22}\}$ generates K as a Hopf algebra. We have $\Delta_K(a_{ii}) = a_{ii} \otimes a_{ii}$ by the coassociativity of ρ . Applying ρ to the equations $x^2y = \beta yx^2, xy^2 = \beta y^2x$ when $A = A(0, \beta)$, we obtain that

$$\begin{aligned} x^2y \otimes a_{11}^2 a_{22} &= \rho(x^2y) = \beta \rho(yx^2) = \beta yx^2 \otimes a_{22} a_{11}^2 = x^2y \otimes a_{22} a_{11}^2 \\ xy^2 \otimes a_{11} a_{22}^2 &= \rho(xy^2) = \beta \rho(y^2x) = \beta y^2x \otimes a_{22}^2 a_{11} = xy^2 \otimes a_{22}^2 a_{11}. \end{aligned}$$

Hence $a_{11}^2 a_{22} = a_{22} a_{11}^2$ and $a_{11} a_{22}^2 = a_{22}^2 a_{11}$. We can similarly get $a_{11}^2 a_{22} = a_{22} a_{11}^2$ and $a_{11} a_{22}^2 = a_{22}^2 a_{11}$ when $A = S(p)$ by applying ρ to the equations $x^2y = pyx^2$ and $xy^2 = -py^2x$. Therefore, K is a group algebra kG where G is a quotient group of $\langle a, b \mid a^2b = ba^2, ab^2 = b^2a \rangle$.

Conversely, we assume that K is a group algebra kG where G is a quotient group of $\langle a, b \mid a^2b = ba^2, ab^2 = b^2a \rangle$. Define a right K -coaction $\rho : A \rightarrow A \otimes K$ by $\rho(x) = x \otimes a$ and $\rho(y) = y \otimes b$. It is easy to check that the corresponding Hopf action on A by $H = K^\circ$ satisfies the Hypothesis 2.1. \square

As a consequence, if A is either $A(\alpha, \beta)$ or $S(p)$ with generic parameters, then A^H is not AS regular for all non-trivial H .

3. HOPF ACTIONS ON A WITH NON-DIAGONALIZABLE μ_A

In this section we will prove Theorems 0.4 and 0.5 for algebras $A(3), A(4), A(5)$. We start with a few lemmas concerning the automorphism $\eta_{\mu_A^\tau}$ of K .

Lemma 3.1. *Suppose K is a finite dimensional Hopf algebra coacting on A via ρ . Let y_{11} be a grouplike element in K .*

- (1) *If $z_{12} \in K$ satisfies $\Delta(z_{12}) = y_{11} \otimes z_{12} + z_{12} \otimes y_{11}$, then $z_{12} = 0$.*

- (2) Suppose $\{v, w\}$ are linearly independent in A and $\rho(v) = v \otimes y_{11}$ and $\rho(w) = w \otimes y_{11} + v \otimes z_{12}$. Then $z_{12} = 0$.

Proof. (1) Let $x = y_{11}^{-1} z_{12}$, then x is a primitive element in K . Since $\text{char } k = 0$ and K is finite dimensional, K does not contain any nonzero primitive element. So $x = 0$ and whence $z_{12} = 0$.

(2) Applying coassociativity, we have $\Delta(z_{12}) = y_{11} \otimes z_{12} + z_{12} \otimes y_{11}$. Then assertion follows from part (1). \square

In the next lemma the global dimension of A could be larger than 2. We say an H -action on A is graded trivial if $H = k\langle\sigma\rangle$ and σ is a graded algebra automorphism of A of the form $\sigma : a \rightarrow \xi^{\deg a} a$ for some root of unity $\xi \in k$.

Proposition 3.2. *Suppose A is an AS regular algebra generated by t_1 and t_2 in degree one such that μ_A is non-diagonalizable. Let H be a Hopf algebra acting on A . Then H is isomorphic to $k\mathbb{Z}/(n)$ for some n and the H -action on A is graded trivial.*

Proof. Up to a base change we may assume that μ_A sends $t_1 \rightarrow at_1$ and $t_2 \rightarrow at_2 + at_1$. So the matrix \mathbb{M} defined by (E2.2.2) is $\begin{pmatrix} a & 0 \\ a & a \end{pmatrix}$. Let $\rho : A \rightarrow A \otimes K$ be the coaction induced by the H -action and write $\rho(t_i) = \sum_{s=1}^2 t_s \otimes y_{si}$ as defined in (E2.2.3). Then the map $f := \eta_{\mu_A^\tau}$ defined in (E2.2.4) sends

$$\begin{aligned} y_{11} &\rightarrow y_{11} + y_{21}, \\ y_{12} &\rightarrow -(y_{11} + y_{21}) + y_{12} + y_{22}, \\ y_{21} &\rightarrow y_{21}, \\ y_{22} &\rightarrow -y_{21} + y_{22}. \end{aligned}$$

It is easy to see that $f^n(y_{11}) = y_{11} + ny_{21}$. Since f has finite order and $\text{char } k = 0$, $y_{21} = 0$. As a consequence, $f(y_{ii}) = y_{ii}$ for $i = 1, 2$. Now we have $f^n(y_{12}) = y_{12} + n(y_{22} - y_{11})$. Since f has finite order and $\text{char } k = 0$, $y_{22} - y_{11} = 0$. Since $y_{22} = y_{11}$, $\Delta(y_{12}) = y_{11} \otimes y_{12} + y_{12} \otimes y_{11}$. By Lemma 3.1(1), $y_{12} = 0$. Thus $K = k\langle y_{11} \rangle$ and $H = k\langle\sigma\rangle$ where σ maps t_i to ξt_i for some root of unity. Thus the H -action on A is graded trivial. \square

Again in the next lemmas, A is AS regular of finite global dimension.

Lemma 3.3. *Suppose A is generated by elements t_1, t_2, t_3 and K -coacts on A with $\rho(t_i) = \sum_{s=1}^3 t_j \otimes y_{ji}$ for all $i = 1, 2, 3$, where $\{y_{ij}\}_{1 \leq i, j \leq 3}$ are elements in K .*

- (1) If $\mu_A : t_1 \rightarrow at_1, t_2 \rightarrow at_2 + bt_1, t_3 \rightarrow a^{-2}t_3$ for some $a, b \in k^\times$, then $\eta_{\mu_A}^\tau$ is determined by

$$\begin{aligned}
 y_{11} &\rightarrow y_{11} + a^{-1}by_{21} \\
 y_{12} &\rightarrow -a^{-1}by_{11} - a^{-2}b^2y_{21} + y_{12} + a^{-1}by_{22} \\
 y_{13} &\rightarrow a^3y_{13} + a^2by_{23} \\
 y_{21} &\rightarrow y_{21} \\
 \eta_{\mu_A}^\tau : \quad y_{22} &\rightarrow -a^{-1}by_{21} + y_{22} \\
 y_{23} &\rightarrow a^3y_{23} \\
 y_{31} &\rightarrow a^{-3}y_{31} \\
 y_{32} &\rightarrow -a^{-4}by_{31} + a^{-3}y_{32} \\
 y_{33} &\rightarrow y_{33}
 \end{aligned}$$

- (2) If $\mu_A : t_1 \rightarrow t_1, t_2 \rightarrow t_2 + bt_1, t_3 \rightarrow t_3 + bt_2 + b^2t_1$ for some $b \in k^\times$, then $\eta_{\mu_A}^\tau$ is determined by

$$\begin{aligned}
 y_{11} &\rightarrow y_{11} + ay_{21} + a^2y_{31} \\
 y_{12} &\rightarrow y_{12} - ay_{11} + ay_{22} - a^2y_{21} + a^2y_{32} - a^3y_{31} \\
 y_{13} &\rightarrow y_{13} - ay_{12} + ay_{23} - a^2y_{22} + a^2y_{33} - a^3y_{32} \\
 y_{21} &\rightarrow y_{21} + ay_{31} \\
 \eta_{\mu_A}^\tau : \quad y_{22} &\rightarrow y_{22} - ay_{21} + ay_{32} - a^2y_{31} \\
 y_{23} &\rightarrow y_{23} - ay_{22} + ay_{33} - a^2y_{32} \\
 y_{31} &\rightarrow y_{31} \\
 y_{32} &\rightarrow y_{32} - ay_{31} \\
 y_{33} &\rightarrow y_{33} - ay_{32}
 \end{aligned}$$

Proof. If necessary, please go back to Section 2 to review the definition of $\eta_{\mu_A}^\tau$.

- (1) In this case, $\mathbb{M} = \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}$. By (E2.2.4), $\eta_{\mu_A}^\tau(y_{ij}) = \mathbb{M}^\tau(y_{ij})(\mathbb{M}^\tau)^{-1}$.

The assertion follows by a direct computation.

- (2) Similar to (1). □

Lemma 3.4. *Retain the hypotheses of Lemma 3.3.*

- (1) If $\eta_{\mu_A}^\tau$ in Lemma 3.3(1) has finite order, then $y_{21} = y_{23} = y_{31} = y_{11} - y_{22} = 0$ and y_{11} and y_{33} are grouplike elements. If a is not a root of unity, then $y_{ij} = 0$ for all $i \neq j$.
- (2) If $\eta_{\mu_A}^\tau$ in Lemma 3.3(1) is the identity and $a^3 \neq 1$, then $y_{ij} = 0$ for all $i \neq j$.
- (3) If $\eta_{\mu_A}^\tau$ in Lemma 3.3(2) has finite order, then $y_{ij} = 0$ and $y_{ii} = y_{jj}$ for all $i \neq j$.

Proof. Let $f = \eta_{\mu_A}^\tau$ in the proof.

- (1) Applying f to y_{11} multiple times, we have $f^n(y_{11}) = y_{11} + na^{-1}by_{21}$. Since f has finite order (and $\text{char } k = 0$), $y_{21} = 0$. As a consequence, $f(y_{11}) = y_{11}$ and

$f(y_{22}) = y_{22}$. Applying f to y_{12} , one sees that $f^n(y_{12}) = y_{12} + na^{-1}b(y_{22} - y_{11})$. Hence $y_{11} = y_{22}$.

Now we have $f(y_{13}) = a^3y_{13} + a^2by_{23}$ and $f(y_{23}) = a^3y_{23}$. So induction shows that $f^n(y_{13}) = a^{3n}y_{13} + (n-1)a^{3n-1}by_{23}$ for all n . Suppose $y_{13} \neq 0$. Then the fact that f has finite order implies that $y_{23} = \alpha y_{13}$. Then equations $f(y_{13}) = a^3y_{13} + a^2by_{23}$ and $f(y_{23}) = a^3y_{23}$ imply that $y_{23} = 0$. If $y_{13} = 0$, then the equation $f(y_{13}) = a^3y_{13} + a^2by_{23}$ implies that $y_{23} = 0$. Therefore $y_{23} = 0$. Similarly, one can show that $y_{31} = 0$.

Using the equations $y_{21} = y_{23} = y_{31} = 0$, one can easily check that $y_{11} = y_{22}$ and y_{33} are grouplike elements.

Finally, if a is not a root of unity, then the finite-orderness of f implies that $y_{13} = y_{32} = 0$. In this case, $\Delta(y_{12}) = y_{11} \otimes y_{11} + y_{11} \otimes y_{12}$. It follows from Lemma 3.1(1) that $y_{12} = 0$. Thus we complete the proof of $y_{ij} = 0$ for all $i \neq j$.

(2) The proof is similar to the proof of part (1).

(3) Note that $f^n(y_{32}) = y_{32} - nay_{31}$ for all n and $na \neq 0$ for all $n > 0$. Since f has finite order, $y_{31} = 0$. Similarly, $y_{32} = 0$. Then $f(y_{21}) = y_{21}$. Applying f to y_{11} , one sees that $y_{21} = 0$. Now $f(y_{ii}) = y_{ii}$ for all i . Since $y_{32} = 0$, $f(y_{23}) = y_{23} + a(y_{33} - y_{22})$. Since f has finite order, we have $y_{33} - y_{22} = 0$. Applying f to y_{12} , one sees that $y_{22} = y_{11}$. Now applying f to y_{22} , one sees that $y_{31} = 0$. Finally applying f to y_{13} , one sees that $y_{12} = y_{23}$. Since $y_{32} = 0$, we have $\Delta(y_{12}) = y_{11} \otimes y_{12} + y_{12} \otimes y_{11}$. By Lemma 3.1(1), $y_{12} = 0$ (and whence $y_{23} = 0$). Finally,

$$\Delta(y_{13}) = y_{11} \otimes y_{13} + y_{12} \otimes y_{23} + y_{13} \otimes y_{33} = y_{11} \otimes y_{13} + y_{13} \otimes y_{11}$$

which implies that $y_{13} = 0$ by Lemma 3.1(1). \square

Going back to global dimension three we have the following.

Proposition 3.5. *Let A be the algebra $A(3)$ or $A(4)$ and let H act on A .*

- (1) *If q is not a root of unity when $A = A(3)$ and p is not a root of unity when $A = A(4)$, then H is a commutative group algebra.*
- (2) *If H is semisimple, then H is a commutative group algebra.*

Proof. (1) By (E1.5.3) and (E1.5.4), the Nakayama automorphism of A is of the form given in Lemma 3.3(1). By Lemma 3.4(1), $y_{ij} = 0$ for all $i \neq j$. So the K -coaction is given by $\rho(t_1) = t_1 \otimes y_{11}$, $\rho(t_2) = t_2 \otimes y_{11}$ and $\rho(t_3) = t_3 \otimes y_{33}$ where each y_{ii} is a grouplike element. So K is a group algebra. Since t_1 and t_3 are skew commutative, y_{11} commutes with y_{33} , see the proof of Proposition 2.11. So K (and whence H) is a commutative group algebra.

(2) Assume H is semisimple. Then A_1 is a direct sum of simples. Let C be the sub-coalgebra generated by $\{y_{ij}\}_{1 \leq i, j \leq 3}$. By Lemma 3.4(1), $y_{21} = y_{23} = y_{31} = y_{11} - y_{22} = 0$. Then every simple comodule is 1-dimensional. Therefore A_1 is a direct sum of three 1-dimensional simple K -comodules, or three 1-dimensional simple H -modules. Since H -action is inner-faithful, H is commutative, or K is cocommutative. Applying Δ to y_{13} and using the cocommutativity, we have $y_{33} = y_{11}$ or $y_{13} = 0$. If $y_{11} = y_{33}$, then Lemma 3.1(1) implies that $y_{13} = 0$. Therefore $y_{13} = 0$. Similarly, $y_{31} = y_{32} = 0$, so $y_{ij} = 0$ for all $i \neq j$. The rest of the proof is similar to the proof of part (1). \square

Up to a k -linear basis change, the Nakayama automorphism of $A(5)$ is of the form given in Lemma 3.3(2). By using Lemma 3.4(3) and the ideas in the proof of Proposition 3.5, we have the following.

Proposition 3.6. *Let A be the algebra $A(5)$ and let H act on A . Then H is a commutative group algebra and the H -action is graded trivial.*

Now we are ready to prove Theorems 0.4 and 0.5.

Proof of Theorem 0.4. (1) This is Proposition 2.8(1).

(2) Proposition 2.9(1).

(3,4) Proposition 3.5(1).

(5) Proposition 3.6.

(6) Proposition 2.13(2). □

Proof of Theorem 0.5. (1) This is Proposition 2.8(2).

(2) Proposition 2.9(2).

(3,4) Proposition 3.5(2).

(6) Proposition 2.13(3).

(7) Proposition 2.14(2). □

Statements in (0.4.8), (0.4.9), (0.5.8) and (0.5.9) are proved in Propositions 2.15 and 2.16.

To conclude this section we prove a lemma which should be useful for the study of Hopf actions on AS regular algebras of higher global dimension. Suppose K coacts on an AS regular algebra A and A is generated by $V := A_1$. Suppose that V is a direct sum of simple K -comodules $\bigoplus_{i=1}^{\alpha} V_i$. We can choose a basis for each V_i , say $v_1^i, v_2^i, \dots, v_{n_i}^i$, for each i . Then K -coaction on V is determined by

$$\rho(v_t^i) = \sum_{s=1}^{n_i} v_s^i \otimes y_{st}^i$$

for all $t = 1, \dots, n_i$. For a fixed i , $\{y_{st}^i\}_{1 \leq s, t \leq n_i}$ is a matrix coalgebra of dimension n_i^2 . Let Y_i be the matrix $(y_{st}^i)_{n_i \times n_i}$, and Y the block matrix $\text{diag}(Y_1, Y_2, \dots, Y_{\alpha})$. Write the Nakayama automorphism of A in terms of the basis $\{v_j^i\}$,

$$\mu_A(v_j^i) = \sum_{i', j'} m_{jj'}^{ii'} v_{j'}^{i'}.$$

Lemma 3.7. *Retain the above notation. Let N be the order of the automorphism $\eta_{\mu_A^r}$.*

- (1) *The matrix \mathbb{M}^N is of the form $\text{diag}(r_1 I_{n_1}, r_2 I_{n_2}, \dots, r_{\alpha} I_{\alpha})$ for some $r_i \in k^{\times}$.*
- (2) *\mathbb{M} is diagonalizable.*
- (3) *If V is a simple H -module, or a simple K -comodule, then \mathbb{M}^N is $r I_n$ for some $r \in k^{\times}$. In this case, the order of μ_A divides $2l \dim_k K$ where l is the AS index of A .*
- (4) *If V is a direct sum of 1-dimensional H -modules, then H is commutative.*

Proof. (1) Since N be the order of the automorphism $\eta_{\mu_A^r}$, \mathbb{M}^N commutes with Y .

The assertion follows from the following fact from linear algebra: if X is a matrix in $M_n(k)$ such that $XY = YX$, then $X = \text{diag}(r_1 I_{n_1}, r_2 I_{n_2}, \dots, r_{\alpha} I_{\alpha})$, for some $r_i \in k^{\times}$.

(2) Since $\text{diag}(r_1 I_{n_1}, r_2 I_{n_2}, \dots, r_\alpha I_\alpha)$ is diagonalizable, so is \mathbb{M} .

(3) The first assertion follows from part (1) by taking $\alpha = 1$. Let g be the automorphism μ_A^N . Then g is graded trivial sending $a \rightarrow r^{\deg a} a$ for all $a \in A$. So $\text{hdet } g = r^l$ [RRZ1, Lemma 5.3(a)]. By [RRZ2, Theorem 5.3], $\text{hdet } \mu_A = 1$, so $\text{hdet } g = 1$. Thus $r^l = 1$. Combining with the fact that N divides $2l \dim_k K$, we obtain the assertion.

(4) Since A_{-1} is a direct sum of 1-dimensional H -module, $H/[H, H]$ acts on A_{-1} . Since the H -action is inner faithful, $[H, H] = 0$ or H is commutative. \square

4. NAKAYAMA AUTOMORPHISM COMMUTES WITH ALL AUTOMORPHISMS

We first briefly recall the definition of Hochschild cohomology. Let M be an A -bimodule, or equivalently, a left A^e -module where $A^e := A \otimes A^{op}$. Consider the cochain complex

(E4.0.1)

$$C_A(M) : 0 \rightarrow M \xrightarrow{d_1} \text{Hom}_k(A, M) \xrightarrow{d_2} \text{Hom}_k(A^{\otimes 2}, M) \xrightarrow{d_3} \text{Hom}_k(A^{\otimes 3}, M) \rightarrow \dots,$$

where $d_n = \sum_{i=0}^n (-1)^i \partial_i$, and for any $f \in \text{Hom}_k(A^{\otimes n-1}, M)$,

$$(\partial_i)(f)(a_1, \dots, a_n) = \begin{cases} a_1 f(a_2, \dots, a_n), & \text{if } i = 0, \\ f(a_1, \dots, a_i a_{i+1}, \dots, a_n) & \text{if } 1 \leq i \leq n-1, \\ f(a_1, \dots, a_{n-1}) a_n, & \text{if } i = n. \end{cases}$$

The n -th Hochschild cohomology of A with coefficients in M is

$$H^n(A, M) = H^n(C_A(M)).$$

As A is free over k , one also has that $H^n(A, M) \cong \text{Ext}_{A^e}^n(A, M)$. A special case is when $M = A^e$ as an A^e -bimodule (and also an A -bimodule). Then $H^n(A, M)$ is a right A^e -module, which can also be viewed naturally as A -bimodules. We need the following lemma.

Lemma 4.1. *Let $\sigma : A \rightarrow B$ be an isomorphism of algebras. Then σ induces a k -linear isomorphism $\Phi^n : H^n(A, A \otimes A^{op}) \rightarrow H^n(B, B \otimes B^{op})$ such that $\Phi^n(a_1 f a_2) = \sigma(a_1) \Phi^n(f) \sigma(a_2)$ for all $a_1, a_2 \in A$.*

Proof. In fact we construct a map at the level of complexes $\Psi : C_A(A \otimes A^{op}) \rightarrow C_B(B \otimes B^{op})$ such that $\Psi^n(a_1 f a_2) = \sigma(a_1) \Psi^n(f) \sigma(a_2)$ for all $a_1, a_2 \in A$. Then the assertion follows from taking homology. For any $f \in \text{Hom}_k(A^{\otimes n}, A \otimes A^{op})$, define $\Psi(f) \in \text{Hom}_k(B^{\otimes n}, B \otimes B^{op})$ by

$$\Psi(f)(b_1, \dots, b_n) = (\sigma \otimes \sigma)(f(\sigma^{-1}(b_1), \dots, \sigma^{-1}(b_n)))$$

where $\sigma \otimes \sigma$ is the corresponding isomorphism from $A \otimes A^{op} \rightarrow B \otimes B^{op}$. One can easily show that Ψ commutes with ∂_i for all i . Hence Ψ commutes with the differential. It also easily follows from the definition that $\Psi(a_1 f a_2) = \sigma(a_1) \Psi(f) \sigma(a_2)$ for all $a_1, a_2 \in A$. The assertion follows by taking homology. \square

As proved in [RRZ1, Lemma 1.2], every Artin-Schelter regular algebra and every Artin-Schelter regular noetherian Hopf algebra has a Nakayama automorphism, see [BrZ] and [RRZ1, Lemma 1.3]. We now show Theorem 0.6. The group of all algebra automorphisms of A is denoted by $\text{Aut}(A)$.

Theorem 4.2. *Let A be an algebra with Nakayama automorphism μ . Let $g \in \text{Aut}(A)$. Then $g\mu g^{-1}$ is also a Nakayama automorphism of A . If A^\times is in the center of A (namely, A has no non-trivial inner automorphisms), then μ commutes with every $g \in \text{Aut}(A)$.*

Proof. Let d and μ be defined as in (E1.1.1). Consider $B = A$ and let $\sigma : A \rightarrow B = A$ be the map g . By Lemma 4.1, $\Phi^d : H^d(A, A \otimes A^{op}) \rightarrow H^d(A, A \otimes A^{op})$ is a k -linear automorphism such that $\Phi^d(a_1 f a_2) = \sigma(a_1) \Phi^d(f) \sigma(a_2)$ for all $a_1, a_2 \in A$. By definition $H^d(A, A \otimes A^{op}) \cong A^\mu$ with a generator u . Then $u' := \Phi^d(u)$ is also an A -bimodule generator. Hence $u' = cu$ for some $c \in A^\times$. Applying Φ^d to the equation $\mu(a)u = ua$, we obtain that $g(\mu(a))u' = u'g(a)$, or equivalently, $g(\mu(g^{-1}(a)))u' = u'a$ for all $a \in A$. Thus $g\mu g^{-1}$ is a Nakayama automorphism of A . \square

Lemma 4.3. *Let A be an algebra with Nakayama automorphism μ_A and such that $A^\times = k^\times$. Suppose that $\{x_1, \dots, x_n\}$ is a set of generators of A such that $\mu_A(x_i) = \lambda_i x_i$ for all i and that the set of the ordered monomials $\{x_1^{a_1} \cdots x_n^{a_n} \mid a_1, \dots, a_n \geq 0\}$ spans the whole algebra A . Assume that λ_1 cannot be written as $\prod_{j>1} \lambda_j^{b_j}$ for any $b_j \geq 0$. Then, for any algebra automorphism g of A , there is a $c \in k^\times$ such that $g(x_1) = cx_1$.*

Proof. Suppose $g(x_1) = \sum c_a x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and write it as

$$g(x_1) = \sum_{a_1 > 0} c_{a^*} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} + \sum_{b_1 = 0} c_{b^*} x_2^{b_2} \cdots x_n^{b_n}$$

where monomials $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\}_{a^*} \cup \{x_2^{b_2} \cdots x_n^{b_n}\}_{b^*}$ appeared in the above expression with nonzero c_{a^*} and c_{b^*} are distinct and linearly independent. Since $\mu_A g \mu_A^{-1} = g$, we have

$$\begin{aligned} \lambda_1^{-1} \left(\sum_{a_1 > 0} c_{a^*} \left(\prod_j \lambda_j^{a_j} \right) x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} + \sum_{b_1 = 0} c_{b^*} \left(\prod_{j>1} \lambda_j^{b_j} \right) x_2^{b_2} \cdots x_n^{b_n} \right) \\ = \sum_{a_1 > 0} c_{a^*} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} + \sum_{b_1 = 0} c_{b^*} x_2^{b_2} \cdots x_n^{b_n}. \end{aligned}$$

Since $\lambda_1 \neq \prod_{j>1} \lambda_j^{b_j}$, all $c_{b^*} = 0$. Thus $g(x_1) = x_1 h$ for some $h \in A$. Similarly, $g^{-1}(x_1) = x_1 f$ for some $f \in A$. Thus $h, f \in A^\times$. Since $A^\times = k^\times$, $g(x_1) = cx_1$ for some $c \in k^\times$. \square

The following proposition is well known. For completeness we give a short proof using Lemma 4.3.

Proposition 4.4. *Let A be $A(1)$ where p_{ij} are generic. Then every algebra automorphism of A preserves the grading of A and $\text{Aut}(A) = (k^\times)^3$.*

Proof. Let g be an algebra automorphism of A .

By (E1.5.1), the Nakayama automorphism of A is of the form

$$\mu_A : t_1 \rightarrow \lambda_1 t_1, t_2 \rightarrow \lambda_2 t_2, t_3 \rightarrow \lambda_3 t_3$$

where $\lambda_1 = p_{21}p_{31}$, $\lambda_2 = p_{12}p_{32}$ and $\lambda_3 = p_{13}p_{23}$. Since p_{ij} 's are generic, λ_1 cannot be written as $\lambda_2^{n_2} \lambda_3^{n_3}$. Applying Lemma 4.3 to $(x_1, x_2, x_3) = (t_1, t_2, t_3)$, $g(t_1) = c_1 t_1$. By symmetry, $g(t_2) = c_2 t_2$ and $g(t_3) = c_3 t_3$ for some $c_i \in k^\times$. The assertion follows. \square

Proposition 4.5. *Let A be $A(2)$ where p is not a root of unity. Then every algebra automorphism of A preserves the grading of A and $\text{Aut}(A) = (k^\times)^2$.*

Proof. By (E1.5.2), the Nakayama automorphism of A is given by

$$\mu : t_1 \rightarrow t_1, \quad t_2 \rightarrow p^{-1}t_2, \quad t_3 \rightarrow pt_3.$$

Let g be any algebra automorphism of A . Applying Lemma 4.3 to $\{x_1, x_2, x_3\} = \{t_2, t_1, t_3\}$, we have that $g(t_2) = ct_2$ for some $c \in k^\times$. Similarly, $g(t_3) = c't_3$ for some $c' \in k^\times$. So $g(t_1^2) = cc't_1^2$. This forces that $g(t_1) = c''t_1$. Therefore g preserves the grading of A .

Rewrite g as $g(t_1) = c_1t_1$, $g(t_2) = c_2t_2$ and $g(t_3) = c_3t_3$. Then $c_3 = c_1^2c_2^{-1}$. Hence $\text{Aut}(A) = k^{\times 2}$. \square

Proposition 4.6. *Let A be $A(6)$ where β is not a root of unity or the algebra $A(7)$ where p is not a root of unity. Then every algebra automorphism of A preserves the grading of A and $\text{Aut}(A) = (k^\times)^2$.*

Proof. The proof for $A(7)$ is very similar to the proof for $A(6)$. So we assume $A = A(6)$.

By (E1.5.6), the Nakayama automorphism of A is given by

$$\mu : x \rightarrow -\beta x, \quad y \rightarrow -\beta^{-1}y.$$

Let g be any algebra automorphism of A . It is well known that A has a k -linear basis $\{x^{n_1}(yx)^{n_2}y^{n_3} \mid n_1 \geq 0\}$. Let $\{x_1, x_2, x_3\} = \{x, yx, y\}$. Then $\lambda_1 = -\beta$, $\lambda_2 = 1$ and $\lambda_3 = -\beta^{-1}$. Since β is not a root of unity, we can apply Lemma 4.3 to this situation. Hence $g(x) = cx$ for some $c \in k^\times$. By symmetry, $g(y) = c'y$ for some $c' \in k^\times$. The assertion follows. \square

Next we deal with non-diagonalizable μ_A , when A is either $A(3)$, or $A(4)$ or $A(5)$. By the way the automorphism group of the Jordan plane

$$k_J[t_1, t_2] := k\langle t_1, t_2 \rangle / ((t_2 + t_1)t_1 = t_1t_2)$$

was given in [Sh]. We will compute the automorphism group of 3-dimensional analogues of $k_J[t_1, t_2]$.

We consider a couple of subgroups. If A is \mathbb{Z} -graded, we use $\text{Aut}_{gr}(A)$ for the subgroup of automorphisms that preserving the \mathbb{Z} -grading. If A is connected graded, an automorphism $g \in \text{Aut}(A)$ is called *unipotent* if

$$g(x) = x + \text{higher degree terms}$$

for all homogeneous element $x \in A$. The subgroup of unipotent automorphisms is denoted by $\text{Aut}_{uni}(A)$. If I is an ideal of A (of codimension 1), let $\text{Aut}(I)$ be the subgroup of $\text{Aut}(A)$ consisting of g preserving I . The following lemma is easy and known and the proof is omitted.

Lemma 4.7. *Let A be a connected graded algebra.*

- (1) *Let x and y be two nonzero elements in A such that $xy = qyx$ for some $1 \neq q \in k$. Then $x, y \in A_{\geq 1}$. As a consequence, $g(x), g(y) \in A_{\geq 1}$ for all $g \in \text{Aut}(A)$.*
- (2) *Let A be generated in degree one. Then $\text{Aut}(A_{\geq 1}) = \text{Aut}_{gr}(A) \ltimes \text{Aut}_{uni}(A)$.*

For most common noncommutative connected graded algebras, $\text{Aut}(A_{\geq 1}) = \text{Aut}(A)$. So the above lemma tells us that we should work on two subgroups $\text{Aut}_{gr}(A)$ and $\text{Aut}_{uni}(A)$. Let ϕ be any k -linear map of A , an element $a \in A$ is called a ϕ -eigenvector (associated to an eigenvalue λ) if $\phi(a) = \lambda a$.

Lemma 4.8. *Let A be an AS regular algebra generated by A_1 and μ_A be the Nakayama automorphism of A . Suppose μ_A has eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ with respect to the basis $\{v_1, \dots, v_n\}$ of A_1 . Assume that $\lambda_i \neq \lambda_j$ for all $i \neq j$, then every graded algebra automorphism of A is of the form*

$$g : v_i \rightarrow c_i v_i$$

for some $c_i \in k^\times$. As a consequence, $\text{Aut}_{gr}(A)$ is a subgroup of $(k^\times)^n$, which is abelian.

Proof. Since g is graded, $g(v_i) \in A_1$. By [RRZ1, Theorem 3.11], $\mu_A g = g \mu_A$. So $g(v_i)$ is a μ_A -eigenvector associated to the eigenvalue λ_i . Since $\lambda_i \neq \lambda_j$ for all $i \neq j$, $g(v_i) = c_i v_i$ for some $c_i \in k$. The assertion follows. \square

Using the similar ideas we have the following.

Lemma 4.9. *Let A be an AS regular algebra generated by $\{t_1, t_2, t_3\}$.*

(1) *Suppose μ_A maps*

$$\mu_A : t_1 \rightarrow at_1, \quad t_2 \rightarrow at_2 + bt_1, \quad t_3 \rightarrow ct_3$$

where $a, b, c \in k^\times$ and $a \neq c$. Then every graded algebra automorphism g is of the form

$$(E4.9.1) \quad g : t_1 \rightarrow c_1 t_1, \quad t_2 \rightarrow c_1 t_2 + c_2 t_1, \quad t_3 \rightarrow c_3 t_3$$

where $c_1, c_3 \in k^\times$ and $c_2 \in k$.

(2) *Suppose μ_A maps*

$$\mu_A : t_1 \rightarrow t_1, \quad t_2 \rightarrow t_2 + b_1 t_1, \quad t_3 \rightarrow t_3 + b_1 t_2 + b_2 t_3$$

where $b_1 \in k^\times$ and $b_2 \in k$. Then every graded algebra automorphism g is of the form

$$(E4.9.2) \quad g : t_1 \rightarrow at_1, \quad t_2 \rightarrow at_2 + c_1 t_1, \quad t_3 \rightarrow at_3 + c_1 t_2 + c_2 t_1$$

where $a \in k^\times$ and $c_1, c_2 \in k$.

(3) *Suppose $q^3 \neq 1$. The group $\text{Aut}_{gr}(A(3))$ consists of all maps of the form (E4.9.1). If G is a finite subgroup of $\text{Aut}_{gr}(A)$, then G is a subgroup of $(k^\times)^2$.*

(4) *Suppose $p^3 \neq 1$. The group $\text{Aut}_{gr}(A(4))$ consists of all maps of the form (E4.9.1). If G is a finite subgroup of $\text{Aut}_{gr}(A)$, then G is a subgroup of $(k^\times)^2$.*

(5) *The group $\text{Aut}_{gr}(A(5))$ consists of all maps of the form (E4.9.2). If G is a finite subgroup of $\text{Aut}_{gr}(A)$, then G is a subgroup of k^\times .*

Proof. Use linear algebra and [RRZ1, Theorem 3.11], namely, $\mu_A g = g \mu_A$. Details are omitted. \square

To prove every algebra automorphism preserves the grading, we need to show that $\text{Aut}_{uni}(A)$ is trivial. The following lemma is useful in computation.

Lemma 4.10. *Let $a, b \in k^\times$ and $c \in k$. Let ϕ be an algebra automorphism of the Jordan plane $k_J[t_1, t_2]$ of the form*

$$\phi : t_1 \rightarrow at_1, \quad t_2 \rightarrow at_2 + bt_1.$$

Then any ϕ -eigenvector in $k_J[t_1, t_2]$ is a polynomial of t_1 .

Proof. Let f be a ϕ -eigenvector. Since ϕ is a graded algebra automorphism, we may assume that f is homogeneous of degree d . Write $f = \sum_{i=0}^d c_i t_1^{d-i} t_2^i \neq 0$. Let s be the integer such that $c_s \neq 0$ and $c_i = 0$ for all $i > s$. It remains to show that $s = 0$. If not, we assume $c_s = 1$ and then $f = t_1^{d-s} t_2^s + c_{s-1} t_1^{d-s+1} t_2^{s-1} + \text{ldt}$ where ldt is (any element) of the form $\sum_{i < s-1} c_i t_1^{d-i} t_2^i$. Since f is a ϕ -eigenvector, we have

$$\begin{aligned} \lambda f &= \phi(f) = (at_1)^{d-s} (at_2 + bt_1)^s + c_{s-1} (at_1)^{d-s+1} (at_2 + bt_1)^{s-1} + \text{ldt} \\ &= a^d t_1^{d-s} t_2^s + s a^{d-1} b t_1^{d-s+1} t_2^{s-1} + a^d c_{s-1} t_1^{d-s+1} t_2^{s-1} + \text{ldt}. \end{aligned}$$

Hence $\lambda = a^d$, $s a^{d-1} b = 0$, which yield a contradiction. Therefore $s = 0$ and the assertion follows. \square

Proposition 4.11. *Let A be an AS regular domain generated by $\{t_1, t_2, t_3\}$. Assume that $\{t_1^{n_1} t_2^{n_2} t_3^{n_3} \mid n_i \geq 0\}$ is a k -linear basis of A and that the subalgebra generated by t_1, t_2 is the Jordan plane with relation $(t_2 + t_1)t_1 = t_1 t_2$ and that $t_3 t_1 = at_1 t_3$ for some $a \in k^\times$.*

(1) *Suppose μ_A maps*

$$\mu_A : t_1 \rightarrow at_1, \quad t_2 \rightarrow at_2 + bt_1, \quad t_3 \rightarrow ct_3$$

where a is not a positive power of c and c is not a positive power of a and both a and c are not roots of unity. Then every unipotent algebra automorphism is the identity on t_1 and t_3 .

(2) *Suppose q is not a root of unity. Then $\text{Aut}(A(3)) = \text{Aut}_{gr}(A(3))$ which consists of all maps of the form (E4.9.1). If G is a finite subgroup of $\text{Aut}(A)$, then G is a subgroup of $(k^\times)^2$.*

(3) *Suppose q is not a root of unity. The center of $A(3)$ is k .*

(4) *Suppose p is not a root of unity. Then $\text{Aut}(A(4)) = \text{Aut}_{gr}(A(4))$ which consists of all maps of the form (E4.9.1). If G is a finite subgroup of $\text{Aut}(A)$, then G is a subgroup of $(k^\times)^2$.*

(5) *Suppose p is not a root of unity. The center of $A(4)$ is k .*

Proof. (1) Since t_3 is a μ_A -eigenvector associated to c , so is $v := g(t_3)$ by Theorem 4.2. Write $v = \sum_{n \geq 0} f_n(t_1, t_2) t_3^n$. Then $\mu_A(v) = \sum_{n \geq 0} \mu_A(f_n) c^n t_3^n$. So each f_i is a μ_A -eigenvector. By Lemma 4.10, $f_0 = w_0 t_1^{n_0}$. So f_0 is a μ_A -eigenvector associated to a^{n_0} . So $c = a^{n_0}$. By the hypothesis, c is not a power of a . So $f_0 = 0$, consequently, $v = h t_3$ for some $h \in A$. Similarly, $g^{-1}(t_3) = h' t_3$. Then h and h' are units and whence $g(t_3) = t_3$.

Since t_1 is a μ_A -eigenvector associated to a , so is $w := g(t_1)$ by Theorem 4.2. Let $w = g(t_1)$ and write $w = \sum_{n \geq 0} f_n(t_1, t_2) t_3^n$ by recycling the notation from the last paragraph. For each n , $f_n t_3^n$ is a μ_A -eigenvector. In particular, f_n is a μ_A -eigenvector. By Lemma 4.10, $f_n = w_n t_1^{d_n}$. So w is generated by t_1 and t_3 . Now we can write $w = \sum t_1^n h_n(t_3)$. Then $h_0(t_3)$ is a μ_A -eigenvector associated to a . If $h_0 \neq 0$, this is impossible as a is not a power of c . So $h_0 = 0$ and $g(t_1) = w = t_1 h$. Similarly, $g^{-1}(t_1) t_1 h' = t_1$. Then h and h' are units and whence $g(t_1) = t_1$.

Now let $u = g(t_2) := t_2 + h$. Since $\mu_A(t_2) = at_2 + bt_1$, one sees that h is a μ_A -eigenvector. So by Lemma 4.10, h is generated by t_1 and t_3 . Applying g to the relation $t_2t_1 = t_1t_2 - t_1^2$, one sees that $ht_1 = t_1h$. By the relation $t_3t_1 = at_1t_3$, we have $h \in k[t_1]_{\geq 2}$ (since g is unipotent).

In the following proof, A is either $A(3)$ or $A(4)$ with parameter not a root of unity.

(2,4) One can easily show that $\text{Aut}(A) = \text{Aut}(A_{\geq 1})$. By Lemma 4.7, it suffices to show that $g \in \text{Aut}_{\text{uni}}(A)$ is the identity. By part (1), $g(t_1) = t_1, g(t_3) = t_3$ and $g(t_2) = t_2 + h$ where $h \in k[t_1]_{\geq 2}$. Then the third relation implies that $h = 0$. The assertion follows.

(3,5) Let f be the center of A . Write $f = \sum_{n \geq 0} f_n t_3^n$ where f_n is in the subalgebra generated by t_1 and t_3 . Since f is a μ_A -eigenvector, by the form of μ_A , each f_n is a μ_A -eigenvector. By Lemma 4.10, $f_n \in k[t_1]$. So f is in the subalgebra generated by t_1 and t_3 . Since we have $t_3t_1 = qt_1t_3$ in $A(3)$ and $t_3t_1 = pt_1t_3$ in $A(4)$, f commutes with both t_1 and t_3 if and only if $f \in k$. Therefore $f \in k$ and the center of A is trivial. \square

5. AUTOMORPHISMS OF $A(5)$

In this section let A be the algebra $A(5)$ defined in the introduction. The relations (E0.3.5) are equivalent to the following relations

$$(E5.0.1) \quad t_i(t_j - t_{j-1}) = t_j(t_i - t_{i-1})$$

for all $1 \leq i < j \leq 3$, where $t_0 = 0$ by convention. We also set $\deg t_i = 1$ for all $i = 1, 2, 3$.

Given a \mathbb{Z} -graded algebra C and a graded algebra automorphism τ of C , the (right) graded twist of C associated to τ , denoted by C^τ , is defined as follows: as a graded k -vector space, $C^\tau = C$, the multiplication $*$ of C^τ is given by

$$(E5.0.2) \quad f * g = f\tau^{\deg f}(g) \in C = C^\tau$$

for all homogeneous elements $f, g \in C = C^\tau$. We use slightly different notation from Section 1. From now on let C be the polynomial ring $k[t_1, t_2, t_3]$.

Lemma 5.1. *The algebra A is (isomorphic to) the graded twist C^σ where σ is the graded algebra automorphism of C sending t_i to $\sum_{j=1}^i t_j$ for all $i = 1, 2, 3$.*

Proof. By definition [Zh] or (E5.0.2), the graded twist C^σ of the commutative ring C , with new multiplication $*$, is generated by t_1, t_2, t_3 and subject to the relations

$$(E5.1.1) \quad t_i * \sigma^{-1}(t_j) = t_i t_j = t_j * \sigma^{-1}(t_i)$$

for all $i < j$. Since $\sigma^{-1}(t_i) = t_i - t_{i-1}$ for all $i = 2, 3$ and $\sigma(t_1) = t_1$. So (E5.1.1) agrees with (E5.0.1). Therefore $A = C^\sigma$. \square

Let ϕ be a k -linear endomorphism of some vector space, say W . An element $f \in W$ is said to be a ϕ -eigenvector if $\phi(f) = cf$ for some $c \in k$, and W is a ϕ -invariant if $\phi(f) = f$, namely f is a ϕ -eigenvector associated to eigenvalue 1.

Lemma 5.2. *Let A and C be defined as above.*

- (1) *Let $f \in C$. Then f is a σ -eigenvector if and only if it is σ -invariant.*
- (2) *$t_1 \in C$ is σ -invariant.*
- (3) *$y_2 := t_2^2 + t_1t_2 - 2t_1t_3 \in C$ is a σ -invariant.*

- (4) If $f \in k[t_1, t_2]$ is σ -invariant, then $f \in k[t_1]$. As a consequence, $k[t_1, t_2]^{(\sigma)} = k[t_1]$.
- (5) Every σ -invariant is generated by t_1 and y_2 . As a consequence, $C^{(\sigma)} = k[t_1, y_2]$.

Proof. (1) Let $V = \bigoplus_{i=1}^3 kt_i$. Then σ is unipotent on V . Since C is generated by V , σ is unipotent on the homogeneous part of C of degree d for any d . Thus, the only eigenvalue of σ on C is 1. So any σ -eigenvector is σ -invariant.

(2) Clear.

(3) By direct computation.

(4) Since σ preserves the degree, we may assume that f is homogeneous. Let $f = \sum_{i=0}^s a_i t_1^{d-i} t_2^i$ where $a_i \in k$ for all $i = 0, \dots, s$ and $a_s \neq 0$ for some $s \leq d$. We claim that $s = 0$. Since f is σ -invariant,

$$\begin{aligned} 0 &= \sigma(f) - f = \sum_{i=0}^s a_i t_1^{d-i} ((t_1 + t_2)^i - t_2^i) \\ &= a_s t_1^{d-s} (s t_1 t_2^{s-1} + \text{ldt}) + \text{ldt} \end{aligned}$$

where ldt means some polynomials of t_2 -degree less than $s-1$. So we have $a_s s = 0$. Since $a_s \neq 0$, $s = 0$ as desired. So $f \in k[t_1]$. The consequence is clear.

(5) First of all, t_1 and y_2 are algebraically independent and elements in $k[t_1, y_2]$ are σ -invariants by parts (2,3).

Now let f be a σ -invariant. Again we may assume that f is homogeneous. Let $g = t_1^d f$ where d is the degree of f . Then g is a polynomial in t_1, t_2 and $t_1 t_3$. Write $t_1 t_3$ as $\frac{1}{2}(t_2^2 + t_1 t_2 - y_2)$. Then g is a polynomial in t_1, t_2, y_2 . Write $g = \sum_{i=0}^d g_i(t_1, t_2) y_2^i$ where $g_i(t_1, t_2)$ is a polynomial in t_1 and t_2 for each i . Since g, y_2 are σ -invariant, by using the fact that t_1, t_2, y_2 are algebraically independent, each $g_i(t_1, t_2)$ is σ -invariant. By part (4), $g_i(t_1, t_2) = c'_i t_1^{d_i}$ where $c'_i \in k$. Since the degree of g is $2d$, we may write $g = \sum_{i=0}^s c_i t_1^{2d-2i} y_2^i$ where $c_i \in k$ and $c_s \neq 0$. Thus $f = \sum_{i=0}^s c_i t_1^{d-2i} y_2^i$ and we need to show that $d-2i \geq 0$ for any i with $c_i \neq 0$. Considering g as a polynomial in t_2 and using the fact $y_2 = t_2^2 + t_1 t_2 - 2t_1 t_3$, the coefficient of the leading term of f is $c_s t_1^{d-2s}$. Thus $d-2s \geq 0$ and f is a polynomial in t_1 and y_2 . \square

Let τ be an algebra automorphism of an algebra B and let f be a nonzero element in B . We say f is τ -normal if $fx = \tau(x)f$ for all $x \in B$. By the relations (E5.0.1) or (E0.3.5), t_1 in A is a σ -normal element.

Lemma 5.3. *Let B be a \mathbb{Z} -graded domain. If $f \in B$ is a τ -normal element for some automorphism τ , then τ is a graded algebra automorphism of B . As a consequence, every nonzero homogeneous component of f is τ -normal.*

Proof. For each element $x \in B$ we can define lower degree $l.\deg(x)$ and upper degree $u.\deg(x)$. Then x is a homogeneous element if and only if $l.\deg(x) = u.\deg(x) = \deg(x)$. Since f is τ -normal, $fx = \tau(x)f$. Then

$$u.\deg(f) + u.\deg(x) = u.\deg(fx) = u.\deg(\tau(x)f) = u.\deg(\tau(x)) + u.\deg(f).$$

Hence $u.\deg(x) = u.\deg(\tau(x))$. Similarly, $l.\deg(x) = l.\deg(\tau(x))$. This implies that x is homogeneous if and only if $\tau(x)$ is homogeneous of the same degree. So the assertion follows. \square

If G is a subgroup of $\text{Aut}(B)$, the fixed subring of B under G -action is denoted by B^G . The following lemma is well known.

Lemma 5.4. *Let B be a graded algebra and τ a graded algebra automorphism. Let G be a subgroup of $\text{Aut}_{gr}(B)$ such that $g\tau = \tau g$ for all $g \in G$. Then G is naturally a subgroup of $\text{Aut}_{gr}(B^\tau)$ and the fixed subring $(B^\tau)^G$ is a graded twist of $(B^G)^\tau$.*

Proof. As a graded vector space $B^\tau = B$. Let \cdot (respectively $*$) be the multiplication of B (respectively, B^τ). Let g be a k -linear graded automorphism of $B = B^\tau$. Since $x*y = x \cdot \tau^{\deg x}(y)$ and since $g \in G$ commutes with τ , g is an algebra endomorphism of B if and only if it is an algebra endomorphism of B^τ . Therefore G is a subgroup of $\text{Aut}_{gr}(B^\tau)$. As a graded vector space, it is clear that $(B^\tau)^G = B^G$. Since τ commutes with G , τ is naturally a graded automorphism of B^G by restriction. By comparing multiplications of $(B^\tau)^G$ and B^G , one sees that $(B^\tau)^G = (B^G)^\tau$ as a graded algebra. \square

Considering $y_2 = t_2^2 + t_1 t_2 - 2t_1 t_3 \in C$ as an element in A , we have

$$y_2 = t_2 * t_2 + 2t_1 * t_2 - 2t_1 * t_3$$

where $*$ is the multiplication in $A = C^\sigma$. When $*$ is omitted,

$$(E5.4.1) \quad y_2 = t_2^2 + 2t_1 t_2 - 2t_1 t_3 \in A.$$

Lemma 5.5. *Let σ be the automorphism of C and A determined by sending t_i to $\sum_{j=1}^i t_j$ for all $i = 1, 2, 3$.*

- (1) *The Nakayama automorphism of A is σ^{-3} .*
- (2) *Fix any integer d , then $f \in A$ is σ^d -invariant, or a σ^d -eigenvector, if and only if $f \in C$ is a σ^d -invariant.*
- (3) *If τ is a graded algebra automorphism of A , then $\tau(t_1) = ct_1$ for some $c \in k^\times$.*
- (4) *If $f \in A$ is a nonzero τ -normal element for some automorphism τ , then $\tau = \sigma^d$ and f is homogeneous of degree d .*
- (5) *An element $f \in A$ is a normal element if and only if f is homogeneous and σ -invariant. As a consequence, normal elements in A are precisely these homogeneous elements in $A^{(\sigma)} = C^{(\sigma)}$.*
- (6) *The subspace kt_1 consists of all possible elements f with the properties: f is τ -normal and $\tau^3 = \mu_A^{-1}$. As a consequence, kt_1 is preserved by any algebra automorphism of A .*
- (7) *Any algebra automorphism of A commutes with σ .*
- (8) *Any algebra automorphism of A becomes a graded algebra automorphism when restricted to $k[t_1, y_2]$.*

Proof. (1) By direct computation, we have $\sigma^3(t_1) = t_1, \sigma^3(t_2) = 3t_1 + t_2$ and $\sigma^3(t_3) = 6t_1 + 3t_2 + t_3$ since $\sigma(t_i) = \sum_{j=1}^i t_j$ for all $i = 1, 2, 3$. Hence $\sigma^3 = \mu_A^{-1}$ by (E1.5.5).

(2) Let G be the subgroup of $\text{Aut}_{gr}(C)$ generated by σ^d . By the proof of Lemma 5.4, $C^G = A^G$. The assertion follows.

(3) This follows from the fact that t_1 is the only normal element in degree 1.

(4) By Lemma 5.3, τ is a graded algebra automorphism and there is a nonzero homogeneous τ -normal component of f , say f_d , of degree d . Then

$$\tau(t_i) * f_d = f_d * t_i = f_d \sigma^d(t_i) = \sigma^d(t_i) f_d = \sigma^d(t_i) * \sigma^{-1}(f_d).$$

Let $i = 1$ in above, we have

$$ct_1 * f_d = \tau(t_1) * f_d = t_1 * \sigma^{-1}(f_d)$$

for some $c \in k^\times$. So f_d is a σ -eigenvector, so a σ -invariant. Using the fact f_d is a σ -invariant, the equation now becomes

$$\tau(t_i) * f_d = \sigma^d(t_i) * f_d$$

or $\tau(t_i) = \sigma^d(t_i)$ for all i .

(5) One implication is part (4). For the other implication, let f be a homogeneous σ -invariant. Then, for all $y \in A = C$,

$$f * y = f \sigma^{\deg f}(y) = \sigma^{\deg f}(y) f = \sigma^{\deg f}(y) * f$$

which implies that f is normal.

(6) If f is τ -normal, by part (4), $\tau = \sigma^d$ for some d and f is homogeneous of degree d . If $\tau^3 = \mu_A^{-1} = \sigma^3$, then $d = 1$ (as σ has infinite order). Hence f has degree 1, and one can easily check that the only degree 1 normal element is kt_1 .

Let ϕ be any algebra automorphism of A and let $f = \phi(t_1)$. Then f is τ -normal with $\tau = \phi\sigma\phi^{-1}$. Since ϕ commutes with $\mu_A^{-1} = \sigma^3$ [Theorem 0.6], we have $\tau^3 = \sigma^3 = \mu_A^{-1}$. By the above paragraph, $f \in kt_1$. Therefore ϕ preserves kt_1 .

(7) Let ϕ be an algebra automorphism of A . By part (6), $\phi(t_1) = ct_1$ for some $c \in k^\times$. Applying ϕ to the equation $t_1x = \sigma(x)t_1$, one obtains that ϕ commutes with σ .

(8) By part (6) any automorphism ϕ sends t_1 to ct_1 . Now y_2 is a σ^2 -normal element. Since ϕ commutes with σ , $\phi(y_2)$ is a σ^2 -normal element. By part (4), $\phi(y_2)$ is homogeneous of degree 2. Thus $\phi(y_2) = \alpha y_2 + \beta t_1^2$ for some $\alpha, \beta \in k$. The assertion follows. \square

Lemma 5.6. *Let g be an algebra automorphism of A such that $g(t_1) = t_1$.*

- (1) *$g(t_2)$ and $g(t_3)$ have zero constant terms.*
- (2) *If g is a graded algebra automorphism, then $g(t_2) = t_2 + at_1$ and $g(t_3) = t_3 + at_2 + dt_1$ for $a, d \in k$.*
- (3) *Let $g(t_2) = t_2 + v$. Then $v \in k[t_1, y_2]$.*

Proof. (1) Applying g to the relation $t_2(t_3 - t_2) = t_3(t_2 - t_1)$, one sees that $g(t_2)$ has zero constant term. Applying g to the relation $t_2(t_3 - t_2) = t_3(t_2 - t_1)$ again, one sees that $g(t_3)$ has zero constant term.

(2) Let $g(t_2) = at_1 + bt_2 + ct_3$ and $g(t_3) = dt_1 + et_2 + ft_3$. Applying g to the relations, we see that $c = 0$, $b = 1$, $f = 1$ and $e = a$. So the assertion follows.

(3) By part (2), $g(t_2) = t_2 + at_1 + f$ where f has lower degree at least 2. Applying g to the relation $t_1(t_2 - t_1) = t_2t_1$, one sees that $t_1f = ft_1$. So f is a σ -invariant. By Lemmas 5.2(5) and 5.4, f is generated by t_1 and y_2 . So $v = at_1 + f \in k[t_1, y_2]$. \square

Now let $\partial : k[t_1, y_2] \rightarrow k[t_1, y_2]$ be the derivation sending f to $(\deg f)f$ for any homogeneous element $f \in k[t_1, y_2]$. We will prove that, given any $u \in k[t_1, y_2]$ and any $\lambda \in k$, the following determines an algebra automorphism of A :

(E5.6.1)

$$\begin{aligned} g(u, \lambda) : \quad & t_1 \rightarrow t_1, \\ & t_2 \rightarrow t_2 + t_1u, \\ & t_3 \rightarrow t_3 + w, \quad \text{where } w = ut_2 + \frac{1}{2}[ut_1 - \partial(u)t_1 + u^2t_1 - \lambda t_1]. \end{aligned}$$

Lemma 5.7. *Let v be an element in $k[t_1, y_2]$.*

- (1) $t_1 v = v t_1$.
- (2) $t_2 v = v t_2 - \partial(v) t_1$.
- (3) $t_3 v = v t_3 - \partial(v) t_2 + \frac{1}{2}(\partial^2 - \partial)(v) t_1$.
- (4) *Suppose g is an algebra endomorphism of A such that $g(t_1) = t_1$ and $g(t_2) = t_2 + v$ where $v \in k[t_1, y_2]$ and $g(t_3) = t_3 + w$ and that $g(y_2) = y_2 + \lambda t_1^2$. Then $v = t_1 u$ where $u \in k[t_1, y_2]$ and $w = u t_2 + \frac{1}{2}[u t_1 - \partial(u) t_1 + u^2 t_1 - \lambda t_1]$. Namely, g is of the form (E5.6.1).*
- (5) *Let g , u and w be as in (E5.6.1) and $v = u t_1$. Then $\sigma^{-1}(w) = w - v$. Assume g is an algebra endomorphism, then g commutes with σ .*
- (6) *Let $g = g(u, \lambda)$ be as in (E5.6.1). Then it is an algebra automorphism with inverse $h := g^{-1}$ given by*

$$\begin{aligned}
 h(t_1) &= t_1 \\
 h(t_2) &= t_2 + u' t_1 \\
 h(t_3) &= t_3 + w', \quad \text{where} \\
 u' &= -u(t_1, y_2 - \lambda t_1^2) \\
 w' &= u' t_2 + \frac{1}{2}[u' t_1 - \partial(u') t_1 + (u')^2 t_1 - (-\lambda) t_1].
 \end{aligned}$$

Using the notation introduced in (E5.6.1), h corresponds to $g(u', -\lambda)$ in (E5.6.1) determined by the parameters $(u', -\lambda)$.

Proof. (1,2,3) By direct computation.

(4) We have

$$\begin{aligned}
 y_2 + \lambda t_1^2 &= g(y_2) = (t_2 + v)^2 + 2t_1(t_2 + v) - 2t_1(t_3 + w) \\
 &= t_2^2 + v t_2 + t_2 v + v^2 + 2t_1 t_2 + 2t_1 v - 2t_1 t_3 - 2t_1 w \\
 &= y_2 + 2v t_2 - \partial(v) t_1 + v^2 + 2t_1 v - 2t_1 w.
 \end{aligned}$$

So $\lambda t_1^2 = 2v t_2 - \partial(v) t_1 + v^2 + 2t_1 v - 2t_1 w$. Thus t_1 divides v . Write $v = t_1 u$. Then $\lambda t_1^2 = 2u t_1 t_2 - \partial(u) t_1^2 + u^2 t_1^2 + u t_1^2 - 2t_1 w$. Hence $w = u t_2 + \frac{1}{2}[u t_1 - \partial(u) t_1 + u^2 t_1 - \lambda t_1]$.

(5) Direct computation.

(6) First we show that g determines an algebra endomorphism of A . Since $u t_1$ commutes with t_1 , g preserves the relation $t_1(t_2 - t_1) = t_2 t_1$. It is easy to see that g preserves the relation $t_1(t_3 - t_2) = t_3 t_1$ if and only if $\sigma^{-1}(w) = w - u t_1$. So, by part (5), g preserves this relation. Next we show that g preserves the relation $t_3(t_2 - t_1) = t_2(t_3 - t_2)$. Let $L = t_3(t_2 - t_1)$ and $R = t_2(t_3 - t_2)$. Write $w = u t_2 + s$ where $s = \frac{1}{2}[u t_1 - \partial(u) t_1 + u^2 t_1 - \lambda t_1]$. Then

$$\begin{aligned}
 g(L) &= (t_3 + w)(t_2 + u t_1 - t_1) = L + w t_2 + w(u t_1 - t_1) + t_3 u t_1 \\
 &= L + u t_2^2 + s t_2 + u t_2(u t_1 - t_1) + s(u t_1 - t_1) \\
 &\quad + (u t_1) t_3 - \partial(u t_1) t_2 + \frac{1}{2}(\partial^2 - \partial)(u t_1) t_1 \\
 &= L + u t_2^2 + s t_2 + u(u t_1 - t_1) t_2 - u \partial(u t_1 - t_1) t_1 + s(u t_1 - t_1) \\
 &\quad + (u t_1) t_3 - \partial(u t_1) t_2 + \frac{1}{2}(\partial^2 - \partial)(u t_1) t_1
 \end{aligned}$$

and

$$\begin{aligned}
g(R) &= (t_2 + ut_1)(t_3 - t_2 + w - ut_1) \\
&= R + t_2(w - ut_1) + ut_1(t_3 - t_2) + ut_1(w - ut_1) \\
&= R + ut_2^2 - \partial(u)t_1t_2 + (s - ut_1)t_2 - \partial(s - ut_1)t_1 \\
&\quad + ut_1(t_3 - t_2) + ut_1(w - ut_1).
\end{aligned}$$

So

$$\begin{aligned}
g(L) - g(R) &= u^2t_1t_2 - u\partial(ut_1 - t_1)t_1 + s(ut_1 - t_1) + \frac{1}{2}(\partial^2 - \partial)(ut_1)t_1 \\
&\quad + \partial(s - ut_1)t_1 - ut_1(ut_2 + s - ut_1) \\
&= -u[\partial(u)t_1 + ut_1 - t_1]t_1 + s(ut_1 - t_1) + \frac{1}{2}[\partial^2(u)t_1 + \partial(u)t_1]t_1 \\
&\quad + \partial(s)t_1 - \partial(u)t_1^2 - ut_1^2 - ut_1s + u^2t_1^2 \\
&= -u\partial(u)t_1^2 + s(ut_1 - t_1) + \frac{1}{2}\partial^2(u)t_1^2 - \frac{1}{2}\partial(u)t_1^2 + \partial(s)t_1 - ut_1s \\
&= -u\partial(u)t_1^2 + \frac{1}{2}[ut_1 - \partial(u)t_1 + u^2t_1 - \lambda t_1](ut_1 - t_1) + \frac{1}{2}\partial^2(u)t_1^2 \\
&\quad + \frac{1}{2}[ut_1 - \partial^2(u)t_1 + (\partial(u)u + u\partial(u))t_1 + u^2t_1 - \lambda t_1]t_1 \\
&\quad - \frac{1}{2}\partial(u)t_1^2 - \frac{1}{2}ut_1[ut_1 - \partial(u)t_1 + u^2t_1 - \lambda t_1] \\
&= 0
\end{aligned}$$

by using the equations $s = \frac{1}{2}[ut_1 - \partial(u)t_1 + u^2t_1 - \lambda t_1]$ and $u\partial(u) = \partial(u)u$. Therefore g is an algebra endomorphism.

One can also check that $g(y_2) = y_2 + \lambda t_1^2$. The map h is of the form g with (u, λ) being replaced by $(u', -\lambda)$. So h is an algebra endomorphism. Similar to the g , we have $h(y_2) = y_2 - \lambda t_1^2$. By the choice of u' , one sees that $h(g(t_2)) = t_2$. So we have $h(g(t_1)) = t_1$, $h(g(t_2)) = t_2$ and $h(g(y_2)) = y_2$. Since $y_2 = t_1^2 - 2t_1t_3$, we also have $h(g(t_3)) = t_3$. Thus h is a left inverse of g . A similar proof shows that g is a left inverse of h . Therefore h is the inverse of g . \square

Theorem 5.8. *Every algebra automorphism τ of $A = A(5)$ is determined by*

$$\begin{aligned}
&t_1 \rightarrow at_1, \\
\tau(a, u, \lambda) : &t_2 \rightarrow a(t_2 + t_1u), \\
&t_3 \rightarrow a(t_3 + w), \quad \text{where } w = ut_2 + \frac{1}{2}[ut_1 - \partial(u)t_1 + u^2t_1 - \lambda t_1].
\end{aligned}$$

for some $a \in k^\times$, $u \in k[t_1, y_2]$ and $\lambda \in k$. Conversely, given any $a \in k^\times$, $u \in k[t_1, y_2]$ and $\lambda \in k$, $\tau(a, u, \lambda)$ is an algebra automorphism of A and the product of two such is determined by

$$\tau(a, u, \lambda)\tau(a', u', \lambda') = \tau(aa', a'u(t_1, y_2) + u'(at_1, a^2(y_2 + \lambda t_1^2)), \lambda + \lambda').$$

Proof. Let τ be an algebra automorphism of A . By Lemma 5.5(6), $\tau(t_1) = at_1$ for some $a \in k^\times$. Replacing τ by $\tau_0 = \xi_{a^{-1}} \circ \tau$, where $\xi_{a^{-1}} : f \rightarrow a^{-\deg f} f$ for any homogeneous element $f \in A$, we obtain $\tau_0(t_1) = t_1$ (or we have $a = 1$ for τ_0). By Lemma 5.6(1), the constant terms of $\tau_0(t_2)$ and $\tau_0(t_3)$ are zero. Hence τ_0 preserves the maximal graded ideal $\mathfrak{m} := A_{\geq 1}$. Since $A = \text{gr } A$ with respect to the \mathfrak{m} -filtration, $\text{gr } \tau_0$ is a graded algebra automorphism of A . By Lemma 5.6(2), $\text{gr } \tau_0$

sends $t_2 \rightarrow t_2 + at_1$ and $t_3 \rightarrow t_3 + at_2 + dt_1$ for some $a, d \in k$. This implies that τ_0 sends $t_2 \rightarrow t_2 + at_1 + hdt$ and $t_3 \rightarrow t_3 + at_2 + dt_1 + hdt$. By Lemma 5.6(3), $\tau_0(t_2) = t_2 + v$ where $v \in k[t_1, y_2]$. By Lemma 5.5(8), τ_0 maps y_2 to $by_2 + \lambda t_1^2$. Consider the t_2^2 term of y_2 , one sees that $\tau_0(y_2) = y_2 + \lambda t_1^2$. By Lemma 5.7(4), $v = ut_1$ and $\tau_0(t_3) = t_3 + w$ where $w = ut_2 + \frac{1}{2}[ut_1 - \partial(u)t_1 + u^2t_1 - \lambda t_1]$, namely, τ_0 is of the form $g(u, \lambda)$ given in (E5.6.1), which is an algebra automorphism by Lemma 5.7(6). Therefore we have $\tau = \xi_a \circ g(u, \lambda)$, which shows the assertion.

Conversely, since $g(u, \lambda)$ is an algebra automorphism, so is $\tau = \xi_a \circ g(u, \lambda)$.

The final statement about the product can be checked directly. \square

6. NAKAYAMA AUTOMORPHISMS AND LOCALLY NILPOTENT DERIVATIONS

In this section we discuss some relationships between the Nakayama automorphism and locally nilpotent derivations. Most of the ideas come from [BeZ].

Recall that a k -linear map $\delta : A \rightarrow A$ is called a derivation if

$$\delta(ab) = a\delta(b) + \delta(a)b$$

for all $a, b \in A$. A derivation δ is called locally nilpotent if, for every $a \in A$, there is an n such that $\delta^n(a) = 0$. The set of derivations (respectively, locally nilpotent derivations) is denoted by $\text{Der}(A)$ (respectively $\text{LND}(A)$). We will deal with a slightly more complicated form.

Definition 6.1. Let A be an algebra.

- (1) A *higher derivation* (or *Hasse-Schmidt derivation*) [HS] on A is a sequence of k -linear endomorphisms $\partial := (\partial_i)_{i \geq 0}$ such that:

$$\partial_0 = \text{id}_A, \quad \text{and} \quad \partial_n(ab) = \sum_{i=0}^n \partial_i(a) \partial_{n-i}(b)$$

for all $a, b \in A$ and for all $n \geq 0$. The set of higher derivations is denoted by $\text{Der}^H(A)$.

- (2) A higher derivation is called *iterative* if $\partial_i \partial_j = \binom{i+j}{i} \partial_{i+j}$ for all $i, j \geq 0$.
(3) A higher derivation is called *locally nilpotent* if
(a) for all $a \in A$ there exists $n \geq 0$ such that $\partial_i(a) = 0$ for all $i \geq n$,
(b) The map $G_{\partial, t} : A[t] \rightarrow A[t]$ defined by

$$(E6.1.1) \quad G_{\partial, t} : a \mapsto \sum_{i \geq 0} \partial_i(a) t^i, \quad t \mapsto t$$

is an algebra automorphism of $A[t]$.

The set of locally nilpotent higher derivations is denoted by $\text{LND}^H(A)$.

- (4) For any $\partial \in \text{Der}^H(A)$, then the kernel of ∂ is defined to be

$$\ker \partial = \bigcap_{i \geq 1} \ker \partial_i.$$

Note that ∂_1 is necessarily a derivation of A . Hence there is a k -linear map $\text{Der}^H(A) \rightarrow \text{Der}(A)$ by sending (∂_i) to ∂_1 . In characteristic 0, the only iterative higher derivation $\partial = (\partial_i)$ on A such that $\partial_1 = \delta$ is given by:

$$(E6.1.2) \quad \partial_n = \frac{\delta^n}{n!}$$

for all $n \geq 0$. It is clear that if δ is a locally nilpotent derivation, then ∂ is a locally nilpotent higher derivation. This iterative higher derivation is called the canonical

higher derivation associated to δ . In this case, we have a map $\text{Der}(A) \rightarrow \text{Der}^H(A)$ sending δ to (∂_i) as defined by (E6.1.2). Hence the map $\text{Der}(A) \rightarrow \text{Der}^H(A)$ is the right inverse of the map $\text{Der}^H(A) \rightarrow \text{Der}(A)$.

The following lemmas are easy.

Lemma 6.2. [BeZ, Lemma 2.2] *Let $\partial := (\partial_i)_{i \geq 0}$ be a higher derivation.*

(1) *Suppose ∂ is locally nilpotent. For any $c \in k$, $G_{c\partial} : A \rightarrow A$ defined by*

$$(E6.2.1) \quad G_{c\partial} : a \rightarrow \sum_{i \geq 0} c^i \partial_i(a)$$

is an algebra automorphism of A .

(2) *If ∂ is iterative and satisfies Definition 6.1(3a), then $G_{\partial,t}$ as defined in (E6.1.1) is an algebra automorphism of A . As a consequence, ∂ is locally nilpotent.*

(3) *If $G : A[t] \rightarrow A[t]$ be a $k[t]$ -algebra automorphism and if $G(a) \equiv a \pmod{t}$ for all $a \in A$, then $G = G_{\partial,t}$ for some $\partial \in \text{LND}^H(A)$.*

Lemma 6.3. *Let A be a connected graded algebra. Let g be a unipotent automorphism of A . For every homogeneous element $a \in A$, define $\partial_i(a)$ to be the $(i + \deg a)$ -degree piece of $g(a)$. Then $\partial := (\partial_i)$ is a locally nilpotent higher derivation and $g = G_{1\partial}$ as defined in (E6.2.1). As a consequence, if $\text{LND}^H(A) = 0$, then $\text{Aut}_{\text{uni}}(A) = \{Id_A\}$.*

We now recall the definition of the Makar-Limanov invariant.

Definition 6.4. Let A be an algebra over k . Let $*$ be either blank or H .

(1) The *Makar-Limanov * invariant* [Ma] of A is defined to be

$$(E6.4.1) \quad \text{ML}^*(A) = \bigcap_{\delta \in \text{LND}^*(A)} \ker(\delta).$$

This means that we have original $\text{ML}(A)$, as well as, $\text{ML}^H(A)$.

(2) We say that A is *LND * -rigid* if $\text{ML}^*(A) = A$, or $\text{LND}^*(A) = \{0\}$.

(3) We say that A is *strong LND * -rigid* if $\text{ML}^*(A[t_1, \dots, t_d]) = A$, for all $d \geq 1$.

By [BeZ, Theorem 3.6], if A is a finitely generated domain of finite GK-dimension and $\text{ML}^H(A) = A$, then A is cancellative. Now we prove Theorem 0.8.

Theorem 6.5. *Let A be an algebra with Nakayama automorphism μ . Suppose that A^\times is in the center of A . Then μ commutes with every locally nilpotent higher derivation $\partial = (\partial_i)$ of A , that is, $\mu\partial_i = \partial_i\mu$ for all i .*

Proof. Let $G_{t,\partial}$ be the automorphism of $A[t]$ defined as in (E6.1.1). Note that $(A[t])^\times = A^\times$ is in the center of $A[t]$. By Theorem 4.2, $\mu_{A[t]}$ commutes with $G_{t,\partial}$. Since $\mu_{A[t]} = \mu_A \otimes Id_{k[t]}$, μ_A commutes with ∂_i for all i . \square

The next lemma is similar to Lemma 4.3.

Lemma 6.6. *Let A be an algebra with Nakayama automorphism μ_A and such that $A^\times = k^\times$. Suppose that $\{x_1, \dots, x_n\}$ is a set of generators of A such that $\mu_A(x_i) = \lambda_i x_i$ for all i and that the set of the ordered monomials $\{x_1^{a_1} \cdots x_n^{a_n} \mid a_1, \dots, a_n \geq 0\}$ spans the whole algebra A . Assume that λ_1 cannot be written as $\prod_{j>1} \lambda_j^{b_j}$ for any $b_j \geq 0$. Then $x_1 \in \text{ML}^H(A[y_1, \dots, y_w])$.*

Proof. Consider the algebra $B := A[y_1, \dots, y_w]$ and $C = B[t]$. Then B and C satisfy the hypotheses of the lemma. Let $\partial = (\partial_i)$ be a locally nilpotent higher derivation of B and consider the algebra automorphism $G_{t,\partial} : B[t] \rightarrow B[t]$. Since $C = B[t]$ satisfies the hypothesis, by Lemma 4.3, $G_{t,\partial}(x_1) = cx_1$ for some $c \in k$. But $G_{t,\partial}(x_1) = x_1 + \sum_{i=1}^{\infty} \partial_i(x_1)t^i$. Thus $c = 1$ and $x_1 \in \ker \partial$. The assertion follows. \square

Definition 6.7. [BeZ, Definition 1.1] Let A be an algebra.

- (a) We call A *cancellative* if $A[y] \cong B[y]$ for some algebra B implies that $A \cong B$.
- (b) We call A *strongly cancellative* if, for any $d \geq 1$,

$$A[y_1, \dots, y_d] \cong B[y_1, \dots, y_d]$$

for some algebra B implies that $A \cong B$.

Now we prove the second half of Corollary 0.7.

Corollary 6.8. *If A is one of the following algebras, then $\text{ML}^H(A[y_1, \dots, y_n]) = A$. As a consequence, A is also strongly cancellative.*

- (1) $A(1)$ where p_{ij} are generic.
- (2) $A(2)$ where p is not a root of unity.
- (3) $A(6)$ where β is not a root of unity.
- (4) $A(7)$ where p is not a root of unity.

Proof. (1) By the proof of Proposition 4.4 for $A = A(1)$ with generic p_{ij} , the hypothesis of Lemma 6.6 was checked for $\{x_1, x_2, x_3\} = \{t_1, t_2, t_3\}$. So $t_1 \in \text{ML}^H(A[y_1, \dots, y_n])$. By symmetry, $t_1, t_2 \in \text{ML}^H(A[y_1, \dots, y_n])$. Hence

$$A \subset \text{ML}^H(A[y_1, \dots, y_n]).$$

It is obvious that

$$A \supset \text{ML}^H(A[y_1, \dots, y_n]).$$

Therefore the assertion follows.

(2) By the proof of Proposition 4.5 for $A = A(2)$ with p not a root of unity, the hypothesis of Lemma 6.6 was checked for $\{x_1, x_2, x_3\} = \{t_2, t_1, t_3\}$. So $t_2 \in \text{ML}^H(A[y_1, \dots, y_n])$. Similarly, $t_3 \in \text{ML}^H(A[y_1, \dots, y_n])$. By the relation $t_1^2 = t_3 t_2 - p t_2 t_3$, we have $t_1^2 \in \text{ML}^H(A[y_1, \dots, y_n])$. A basic property of locally nilpotent derivation implies that $t_1 \in \text{ML}^H(A[y_1, \dots, y_n])$. The assertion follows.

(3,4) The assertion follows by using a similar idea as above and the fact proved in Proposition 4.6. \square

Now we are ready to prove Corollary 0.9.

Proof of Corollary 0.9. For algebras $A(1), A(2), A(6)$ and $A(7)$, see Corollary 6.8. For algebras $A(3), A(4)$ and $A(5)$, we have seen that the center is trivial [Lemma 4.10(3) and Proposition 4.11(3,5)]. By [BeZ, Proposition 1.3], these algebras are cancellative. \square

We use the following diagram to illustrate the ideas involved.

$$\begin{array}{ccccc}
 \mu_A & & & & \text{ZCP} \\
 \downarrow \text{controls} & & & & \uparrow \text{answers} \\
 \text{Aut}(A) \xleftarrow{\cong} \text{Aut}_{uni}(A) & \longrightarrow & \text{LND}^H(A[y_1, \dots, y_n]) & \longrightarrow & \text{ML}^H(A[y_1, \dots, y_n])
 \end{array}$$

In the paper [BeZ], the authors use similar ideas with μ_A being replaced by the discriminant.

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